

For Reference

NOT TO BE TAKEN FROM THIS ROOM

For Reference

NOT TO BE TAKEN FROM THIS ROOM

Ex LIBRIS
UNIVERSITATIS
ALBERTAENSIS



THE UNIVERSITY OF ALBERTA

POLYADIC POST ALGEBRAS

by



VERNON JOSEPH WILLIS

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY.

DEPARTMENT MATHEMATICS.

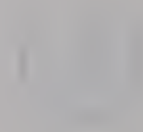
EDMONTON, ALBERTA

DATE

CHICAGO, ILL., U.S.A.

1954

CHICAGO, ILL., U.S.A.



CHICAGO, ILL., U.S.A.

CHICAGO, ILL., U.S.A.
CHICAGO, ILL., U.S.A.
CHICAGO, ILL., U.S.A.

CHICAGO, ILL., U.S.A.

CHICAGO, ILL., U.S.A.

CHICAGO, ILL., U.S.A.

UNIVERSITY OF ALBERTA
FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read,
and recommended to the Faculty of Graduate Studies for
acceptance, a thesis entitled POLADIC POST ALGEBRAS
submitted by VERNON JOSEPH WILLIS in partial fulfilment
of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

The thesis generalizes the concepts introduced by P. Halmos in his work on Polyadic Boolean algebra. An operator called a Post existential quantifier was defined on a Post algebra; from which, it was shown that these operators are determined by the Boolean existential quantifier formed by the restriction of the operator on the base algebra of the Post algebra and by the stationary distinguished elements that result under the given operator. With the Post existential quantifier and the Post algebra considered as a monadic Post algebra, a study of the concrete example of a functional monadic algebra and of the constant on a monadic Post algebra was carried out. The simple monadic Boolean algebra, which is of importance in the theory of monadic Boolean algebras, was replaced by the concept of a basically monadic simple monadic Post algebra which was found to be basic for the theory of monadic Post algebras.

Consideration of the general theory of polyadic Post algebras, showed that for the cases that would be of interest in logic, the theory was determined by the resulting polyadic Boolean algebra formed by the restriction of the operators of the polyadic Post algebra on the base algebra of the given Post algebra.

Linking the theory of polyadic Post algebras with logic resulted in the definition and consideration of a polyadic logic, of a model, of being semantically complete and consistent. With these concepts it was found that a necessary and sufficient condition for the polyadic Post logic to be semantically complete was that the ideal used in the definition of the logic should be a polyadic Post ideal.

LIST OF CONTENTS

	Page
Introduction.	3
Chapter I. Post Algebras.	5
Chapter II. Monadic Post Algebras.	12
1. Post existential and universal quantifiers.	12
2. Functional monadic Post algebras.	22
3. Constants on a monadic Post algebra.	27
4. Basically monadic simple monadic Post algebras.	31
Chapter III. Polyadic Post Algebras.	37
1. Quantifier and transformation Post algebras.	37
2. Polyadic Post algebras.	40
Chapter IV. Polyadic Ideals and Logics.	52
1. Polyadic ideals.	52
2. Polyadic Post logics.	57
References.	66

INTRODUCTION

In recent years it has become in fashion to exploit the algebraic structures inherent in the logical systems in order to study the logical. ([3], [6] to [9]) An approach in this direction by Rasowia and Sikorski [9] has been to utilize the usual categories of algebraic structures and their morphisms such as the Boolean algebras and the Boolean homomorphisms to carry out this study. However this approach did not result in any algebraic object that could be considered as an algebraic counterpart of such logical objects as the lower predicate calculus. As such an object was deemed appropriate Halmos [6] introduced the theory of the polyadic Boolean algebras which were to be the algebraic counterpart of the lower predicate calculi, and Tarski [7] introduced the concept of the cylindrical Boolean algebras for the lower predicate calculi with equality. These developments led to an approach which could be applied to other logical systems.

For the classical propositional and classical lower predicate calculi it has been possible to generalize syntactically or from a formal point of view these logical systems to the n -valued propositional and n -valued lower predicate calculi [11], resulting in the question as to what are the algebraic structures related to these logical systems. For the n -valued propositional calculi, Rosenbloom [10] introduced the Post algebras which have recently been studied by several authors. ([4], [5], [13], [14]) As to the algebraic structures related to the n -valued lower predicate calculi nothing is known. It is to this unknown that this thesis is directed. The approach will not be to derive the

appropriate algebraic object directly from the logical but rather by an appropriate generalization of Halmos's polyadic Boolean algebra to arrive at an algebraic object which may or may not be satisfactory as the algebraic mirroring of the n -valued predicate calculus. The satisfactoriness will not be considered and indeed the satisfactoriness of the Post algebra with respect to the n -valued propositional calculus, which has not been sufficiently ~~analyzed~~, ^{analyzed,} will not be considered.

This thesis which follows the work of P. Halmos [6] shall be in three parts:

- (1) generalization of the concepts of the monadic Boolean algebras and a discussion of such,
- (2) generalization of the concepts of the polyadic Boolean algebras and a discussion of such,
- (3) the introduction of the polyadic Post logics.

CHAPTER I

POST ALGEBRAS

This chapter will list the basic definitions and results pertaining to Post algebras, that will be utilized in the later chapters. These ideas have evolved through the investigations of several authors.

(Rosenbloom [10], Epstein [5], Trackzyk [13], [14] and Dwinger [4])

Firstly two equivalent definitions of Post algebras will be given of which the second will be of greater use. Note that in what follows, the following simplification will be made: $a \wedge b = ab$.

Definition 1.1. A Post algebra of order n , $n \geq 2$, is a bounded distributive lattice P satisfying the following conditions:

(1) there exists an ascending sequence of elements of P :

$0 = e_0 \leq e_1 \leq \dots \leq e_{n-1} = 1$, $0 \neq 1$, such that every element a of

P can be written as $\bigcup_{i=1}^{n-1} a_i e_i$ where for every i , $1 \leq i \leq n-1$,

a_i is an element of the Boolean algebra B of complemented elements of P ,

(2) if $a \in B$ and $ae_i \leq e_{i-1}$ for some i , $1 \leq i \leq n-1$, then $a = 0$.

Definition 1.2. A Post algebra of order n , $n \geq 2$, is a bounded distributive lattice P satisfying the following condition:

There exists an ascending sequence of elements of P :

$0 = e_0 \leq e_1 \leq \dots \leq e_{n-1} = 1$, $0 \neq 1$ such that every element $a \in P$ has

a unique representation $a = \bigcup_{i=1}^{n-1} a_i e_i$ where a_i belongs to the set of complemented elements of P and $a_1 \geq a_2 \geq \dots \geq a_{n-1}$.

The elements $e_i (i=0, \dots, n-1)$ of P with the properties stated in Def. 1.1. have the properties stated in Def. 1.2. and vice versa. Denote the algebra P by $\langle e_0, \dots, e_{n-1} : B \rangle$. The unique representation mentioned in Def. 1.2. will be called the monotonic representation of a . From the definition it follows that only e_0, e_{n-1} of $\{e_i : i = 0, \dots, n-1\}$ belong to B and the elements of $\{e_i : i = 0, \dots, n-1\}$ are distinct.

Given a Post algebra $P = \langle e_0, \dots, e_{n-1} : B \rangle$ it follows that B is uniquely determined as it is the set of complemented elements of P but it is not obvious that the elements $\{e_i : i = 0, \dots, n-1\}$ are uniquely determined. Dwinger [4] was able to answer this question.

Theorem 1.3. Let P be a Post algebra and let $P = \langle e_0, \dots, e_{n-1} : B \rangle = \langle e'_0, \dots, e'_{m-1} : B \rangle$ then $m = n$ and $e_i = e'_i$ for $1 \leq i \leq n-1$.

Call the elements $\{e_i : i = 0, \dots, n-1\}$ the distinguished elements of $P = \langle e_0, \dots, e_{n-1} : B \rangle$ and B the base algebra of P .

Since each $a \in P$ has a unique monotonic representation, it is possible to define a set of functions $d_i : P \rightarrow B$ ($i = 1, \dots, n-1$) by

the monotonic representation $a = \bigcup_{i=1}^{n-1} d_i(a) e_i$. The following facts

on the mappings d_i will be utilized in the later chapters:

- (1) for all $a \in B$, $d_i(a) = a$,
- (2) if $j \leq i$ then $d_j(e_i) = 1$, and
- (3) if $j > i$ then $d_j(e_i) = 0$.

In the investigations dealing with the structure of the Post algebras, it has been demonstrated that the base algebra plays an important part. This is reflected in the next two results proved by Epstein [5].

Theorem 1.4.

- (1) If $a = \bigcup_{j \in I}^P a_j$, where $I \neq \emptyset$, exists and $a_j \in P$ for all $j \in I$ then $\bigcup_{j \in I}^B d_i(a_j)$ exists and $d_i(a) = \bigcup_{j \in I}^B d_i(a_j)$ for each $i = 1, \dots, n-1$.
- (2) If $\bigcup_{j \in I}^B d_i(a_j)$ exists for each $i = 1, \dots, n-1$ where $a_j \in P$ for all $j \in I \neq \emptyset$ then $\bigcup_{j \in I}^P a_j$ exists and is equal to the element a which is determined by $d_i(a) = \bigcup_{j \in I}^B d_i(a_j)$.

The statement arising by replacing $\bigcup$ by $\bigcap$ in the above also holds.

The following special cases of theorem 1.4, which can also be easily demonstrated, should be noted:

- (1) $a \vee b = \bigcup_{i=1}^{n-1} (d_i(a) \vee d_i(b))e_i$,
- (2) $a \wedge b = \bigcup_{i=1}^{n-1} (d_i(a) \wedge d_i(b))e_i$, and
- (3) $a \leq b$ iff $d_i(a) \leq d_i(b)$ for all i .

Theorem 1.5. If $a_i \in B$ and either of the $\bigcup_{i \in I}^B a_i$, $\bigcup_{i \in I}^P a_i$ exist, then the other exists and the two are equal.

Since the Post algebra in its definition possesses statements that relate each element to various distinguished elements of the algebra, the various concepts of an algebra such as homomorphism and subalgebra must be analysed to a greater extent. Trackzyk [13] gave the definition of a Post homomorphism, and Dwinger [4] the definition of a Post subalgebra.

Definition 1.6. Let $P = \langle e_0, \dots, e_{n-1} : B \rangle$ and $P' = \langle e'_0, \dots, e'_{m-1} : B' \rangle$.

I. A map $h: P \rightarrow P'$ is a Post homomorphism if:

(1) h is a lattice homomorphism, and

(2) $h(e_i) \in \{e'_j : j = 0, \dots, m-1\}$ for $0 \leq i \leq n-1$ and $h(0) = 0$

and $h(1) = 1$.

II. A Post homomorphism h is natural if $h(e_i) \neq h(e_k)$ for $i, k = 0, \dots, n-1$, $i \neq k$.

Definition 1.7. Let $P = \langle e_0, \dots, e_{n-1} : B \rangle$ be a Post algebra. A subset P' of P is a Post subalgebra of P iff:

(1) P' is a sublattice of P containing 0 and 1, and

(2) if B' is the Boolean algebra of complemented elements of P'

and if $0 = e_{k_0} \leq e_{k_1} \leq \dots \leq e_{k_p} = e_{n-1}$, where $k_0 < k_1 < \dots < k_p$ are those elements of the set $\{e_i : i = 0, \dots, n-1\}$ which belong to P' then any

element a of P' can be expressed as $a = \bigcup_{i=1}^p a_i e_{k_i}$ where $a_i \in B'$.

Clearly B' is a subalgebra of B and P' is a Post algebra of order $p+1$ with $e_{k_0}, e_{k_1}, \dots, e_{k_p}$ being the distinguished elements of P'

because the conditions (1) and (2) of def. 1.1. are satisfied.

Also every element of the form $\bigcup_{i=1}^p a_i e_{k_i}$ where $a_i \in B'$ belongs to P' ,

since P' is a sublattice of P . Notice also that $B' = P' \cap B$.

With these definitions of Post homomorphism and Post subalgebra the following results follow.

Theorem 1.8. Let P and P' be Post algebras and let $h: P \rightarrow P'$ be a Post homomorphism, then $h(P)$ is a Post subalgebra of P' .

Theorem 1.9. If $P = \langle e_0, \dots, e_{n-1} : B \rangle$ and $P' = \langle e'_0, \dots, e'_{n-1} : B' \rangle$ are Post algebras, A is a Boolean subalgebra of B' and h_0 is an isomorphism of B onto A then the mapping h defined by $h(a) =$

$$= \bigcup_{i=1}^{n-1} h_0(d_i(a)) e'_i \text{ for } a \in P \text{ is an isomorphism of } P \text{ onto}$$

$$\langle e'_0, \dots, e'_{n-1} : A \rangle.$$

Theorem 1.10. If P' is a Post subalgebra of $P = \langle e_0, \dots, e_{n-1} : B \rangle$ and e_{k_j} ($j = 0, \dots, m$) are the distinguished elements of P contained

in P' then for all $a \in P'$, $\bigcup_{j=1}^m d_{k_{j-1}} + 1(a) e_{k_j}$ is the monotonic

representation of a with respect to P' .

A Post algebra being a distributive lattice, the concept of an ideal still has a sense. However, not all ideals are ~~kernels~~ ^{kernels} of Post homomorphisms. In order to characterize those ideals which are ~~kernels~~ ^{kernels,} Dwinger [4] introduced the following concept.

Definition 1.11. Let $P = \langle e_0, \dots, e_{n-1} : B \rangle$ be a Post algebra. Suppose that Δ is a subset of P , then Δ is called a Post ideal of P , if there exists an integer k , $1 \leq k \leq n-1$, and an ideal Δ' of B such that an element a of P belongs to Δ iff $d_i(a) \in \Delta'$ for $i \geq k$.

This automatically implies that Δ is an ideal of P . Denote Δ by (Δ', k) . The following facts follow directly from the definition:

- (1) $\Delta \cap B = \Delta'$,
- (2) Δ is proper if Δ' is proper, and
- (3) if (Δ', k) is proper then $e_i \in \Delta$ iff $i < k$.

Definition 1.12. The order of a proper lattice ideal Δ of a Post algebra of order n is the smallest integer k of $\{0, \dots, n-1\}$ such that $e_k \notin \Delta$.

Theorem 1.13. If $\Delta = (\Delta', k)$ is proper then the number k is the order of Δ .

Although not every ideal is a Post ideal one can demonstrate that a prime ideal is a Post ideal. Dwinger [4] using the concept of a Post ideal and a Post homomorphism was able to prove the following result.

Theorem 1.14. Let $P = \langle e_0, \dots, e_{n-1} : B \rangle$ be a Post algebra. A proper ideal of P is the ~~kernel~~^{kernel} of a Post homomorphism iff it is a Post ideal.

CHAPTER II

MONADIC POST ALGEBRAS

The development of this chapter will draw heavily on Paul Halmos's work on monadic Boolean algebras. Indeed the procedure will be to generalize the concepts introduced by Halmos and to study these newly introduced concepts.

1. Post Existential and Universal Quantifiers.

Let $P = \langle e_0, \dots, e_{n-1}; B \rangle$ be a Post algebra.

Definition 2.1. An operator $\exists: P \rightarrow P$ is a Post existential quantifier iff

- (1) $\exists$ is a closure operator, and
- (2) $\exists P$ is a Post subalgebra.

Since a Post algebra of order 2 is a Boolean algebra the definition of a Post existential quantifier includes the definition of a Boolean existential quantifier.

Theorem 2.2. $\exists B = B \cap \exists P$.

Proof. Let $a \in B \cap \exists P$. Since $\exists$ is a closure operator, $a \in \exists P$ iff $\exists a = a$. As a also belongs to B , $a = \exists a \in \exists B$, and so $B \cap \exists P \subset \exists B$.

It remains to show that $\exists B \subset B \cap \exists P$. Since $\exists B \subset \exists P$ this will

follow if one can demonstrate that $\exists B \subset B$. Assume that $\exists B \not\subset B$ which means that there exists a $b \in B$ such that $\exists b \notin B$. As $\exists P$ is a

Post subalgebra of P it follows from def. 1.7 and the following

monotonic representation^{se} of remarks that $\exists b$ has the form $\bigcup_{j=1}^m b_j e_{k_j}$ where the b_j belong to

the complemented elements of $\exists P$. Now there will exist integers ℓ, k where $\ell < k$ and $b_k < b_\ell$ for if not then the b_j would all be equal and $\exists b$ would be equal to b_m and so would belong to B . Noticing

that $\bigcup_{j=1}^m b_j e_{k_j}$ can be expressed in the form $\bigcup_{i=1}^{n-1} a_i e_i$ where $a_i = b_j$ for $k_{j-1} < i \leq k_j$ and that $b = \bigcup_{i=1}^{n-1} b e_i$ it follows from $\exists b \geq b$ that

$b_j \geq b$ for all $j = 1, \dots, m$. In particular we have $b_k \geq b$. Since $\exists$ is a closure operator this results in $\exists(b_k) \geq \exists(b)$, and since $b_k \in \exists P$ the inequality becomes $b_k \geq \exists b$. Considering again the monotonic representation of b_k and $\exists b$ in P , one will find that $b_k \geq b_j$ for $j = 1, \dots, m$ and in particular that $b_k \geq b_\ell$ which is contrary to the choice of b_k and b_ℓ . Hence we have $\exists B \subset B$ and the desired proof will follow.

From the theory of Post algebras we know that $B \cap \exists P$ is the base algebra of P , which means that $\exists B$ is the base algebra of $\exists P$. Let e_{k_j} ($j = 0, \dots, m$) be the distinguished elements of $\exists P$. This gives

$$\exists P = \langle e_{k_0}, \dots, e_{k_1}, \dots, e_{k_m} : \exists B \rangle.$$

$\exists B$ being a subalgebra of B and $\exists$ being a closure operator results in $\exists$ restricted to B being a Boolean existential quantifier.

The question arises as to whether a concrete example of a non-trivial Post existential quantifier can be found. This was answered in the affirmative by Halmos [6] for the case of a Boolean algebra.

Theorem 2.3. If $P = \langle e_0, \dots, e_{n-1}; B \rangle$ is a Post algebra and $\mathbb{E}$ is a Boolean existential quantifier on B , then there exists a Post existential quantifier on P which is an extension of $\mathbb{E}$.

Proof. From $\{0, \dots, n-1\}$ choose a sequence $0 = k_0 < k_1 < \dots < k_m = n-1$.

Define an operator $\mathbb{E}^*$ on P as follows: $\mathbb{E}^*a =$

$$= \bigcup_{j=1}^m \mathbb{E} d_{k_{j-1}+1}(a) e_{k_j} \quad \text{for all } a \in P. \quad \text{Now for all } a \in B,$$

$$\mathbb{E}^*a = \bigcup_{j=1}^m \mathbb{E} d_{k_{j-1}+1}(a) e_{k_j} = \bigcup_{j=1}^m \mathbb{E} a e_{k_j} = \mathbb{E}a \quad \text{which means that } \mathbb{E}^* \text{ is an}$$

extension of $\mathbb{E}$. It remains to be shown that $\mathbb{E}^*$ is a Post existential quantifier. Let us show first that $\mathbb{E}^*$ is a closure operator.

(1) From definition of $\mathbb{E}^*$, $\mathbb{E}^*0 = \mathbb{E}0 = 0$.

(2) Since the $d_i(a)$ ($i = 1, \dots, n-1$) are monotonic decreasing it follows that for $k_{j-1} < i \leq k_j$, $d_{k_{j-1}+1}(a) \geq d_i(a)$ implying that $\mathbb{E} d_{k_{j-1}+1}(a) \geq d_i(a)$. Hence $\mathbb{E}^*a \geq a$ will follow after taking the proper

meets and joins.

(3) Consider now $\mathbb{E}^*\mathbb{E}^*a$.

Since $\mathbb{E}^*a = \bigcup_{j=1}^m \mathbb{E} d_{k_{j-1}+1}(a) e_{k_j}$ it follows that $d_i(\mathbb{E}^*a) = \mathbb{E} d_{k_{j-1}+1}(a)$

for $k_{j-1} < i \leq k_j$. Hence $d_{k_{j-1}+1}(\mathbb{E}^*a) = \mathbb{E} d_{k_{j-1}+1}(a)$ and

$$d_i(E^*E^*a) = E d_{k_{j-1}+1}(E^*a) = E E d_{k_{j-1}+1}(a) = E d_{k_{j-1}+1}(a) = d_i(E^*a) \quad \text{for}$$

$k_{j-1} < i \leq k_j$ and it follows that $E^*E^*a = E^*a$.

$$\begin{aligned} (4) \quad \text{Now } E^*(a \vee b) &= \bigcup_{j=1}^m E d_{k_{j-1}+1}(a \vee b) e_{k_j} = \\ &= \bigcup_{j=1}^m E(d_{k_{j-1}+1}(a) \vee d_{k_{j-1}+1}(b)) e_{k_j} = \bigcup_{j=1}^m (E d_{k_{j-1}+1}(a) \vee E d_{k_{j-1}+1}(b)) e_{k_j} = \\ &= \bigcup_{j=1}^m E d_{k_{j-1}+1}(a) e_{k_j} \vee \bigcup_{j=1}^m E d_{k_{j-1}+1}(b) e_{k_j} = E^*a \vee E^*b. \end{aligned}$$

By 1-4 we have that E^* is a closure operator. Next it is to be shown that E^*P is a Post subalgebra.

Since $E^*0 = E0 = 0$ and $E^*1 = E1 = 1$, $0, 1 \in E^*P$. Let $a, b \in E^*P$ which means there exist $p, g \in P$ such that $E^*p = a$ and $E^*g = b$. As $a \vee b = E^*p \vee E^*g = E^*(p \vee g) \in E^*P$ it follows that E^*P is closed with

$$\begin{aligned} \text{respect to } "\vee". \quad \text{Now } E^*(a \wedge b) &= \bigcup_{j=1}^m E d_{k_{j-1}+1}(a \wedge b) e_{k_j} = \\ &= \bigcup_{j=1}^m E(d_{k_{j-1}+1}(a) \wedge d_{k_{j-1}+1}(b)) e_{k_j} = \bigcup_{j=1}^m E(d_{k_{j-1}+1}(E^*p) \wedge d_{k_{j-1}+1}(E^*g)) e_{k_j} = \\ &= \bigcup_{j=1}^m E(E d_{k_{j-1}+1}(p) \wedge E d_{k_{j-1}+1}(g)) e_{k_j} = \bigcup_{j=1}^m (E d_{k_{j-1}+1}(p) \wedge E d_{k_{j-1}+1}(g)) e_{k_j} = \end{aligned}$$

(since it is known for a closure operator E that

$$E(Ea \wedge Eb) = (Ea \wedge Eb) = E^*p \wedge E^*g = a \wedge b. \quad \text{Hence } a \wedge b \in E^*P$$

and E^*P is closed with respect to $"\wedge"$. Since E_B is a subalgebra of B

and $E_B \subset E^*P$, E_B is contained in the set of complemented elements of E^*P .

As $\mathbb{E}^*P$ is a lattice subalgebra containing 0 and 1 a complemented element of $\mathbb{E}^*P$ must belong to B and therefore to $\mathbb{E}^*B$. It follows that $\mathbb{E}B$ is the set of all elements of $\mathbb{E}^*P$ complemented with respect to $\mathbb{E}^*P$. From the construction of the operator $\mathbb{E}^*$ it is immediate that if $e_{k_{j-1}} < e_i \leq e_{k_j}$ then $\mathbb{E}^*e_i = e_{k_j}$ which gives $\{e_i : i = 0, \dots, n-1\} \cap \mathbb{E}^*P = \{e_{k_j} : j = 0, \dots, m\}$. Because every $a \in \mathbb{E}^*P$ can be expressed as $a = \mathbb{E}^*P = \bigcup_{j=1}^m \mathbb{E}d_{k_{j-1}+1}^{(p)} e_{k_j}$ the second condition of a Post subalgebra holds and $\mathbb{E}^*P$ will be a Post subalgebra of P .

~~The third condition for $\mathbb{E}^*$ to be a Post existential quantifier holds since $\mathbb{E}^*$ is an extension of $\mathbb{E}$ and $\mathbb{E}B \subseteq B$.~~

Therefore a Boolean existential quantifier may be extended to a Post quantifier in a not necessarily unique fashion, indeed, it will be demonstrated that all Post quantifiers are extensions of this sort.

The behaviour of $\mathbb{E}$ on the distinguished elements will be determined by the following theorem.

Theorem 2.4. If $P = \langle e_0, \dots, e_{n-1} : B \rangle$ is a Post algebra and $\mathbb{E}$ is a Post existential quantifier on P such that $\mathbb{E}P = \langle e_{k_0}, \dots, e_{k_m} : \mathbb{E}B \rangle$ then

$k_{l-1} < \alpha \leq k_l$ implies that $\mathbb{E}e_\alpha = e_{k_l}$.

Proof. Since $\mathbb{E}P$ is a Post subalgebra, $\mathbb{E}e_\alpha = \bigcup_{j=1}^m a_j e_{k_j}$ where the a_j 's form a decreasing sequence of elements of $B \cap \mathbb{E}P$. Hence $d_i(\mathbb{E}e_\alpha) = a_\ell$ for $k_{\ell-1} < i \leq k_\ell$ and especially $d_\alpha(\mathbb{E}e_\alpha) = a_\ell$. As $d_\alpha(\mathbb{E}e_\alpha) \geq d_\alpha(e_\alpha) = 1$ one has that $a_\ell = 1$ and so $\mathbb{E}e_\alpha \geq e_{k_\ell}$. Now $\alpha \leq k_\ell$ implies that $\mathbb{E}e_\alpha \leq \mathbb{E}e_{k_\ell}$ since $\mathbb{E}$ is a closure operator and since $\mathbb{E}e_{k_\ell} = e_{k_\ell}$ it follows that $\mathbb{E}e_\alpha \leq e_{k_\ell}$ and so $\mathbb{E}e_\alpha = e_{k_\ell}$.

The distinguished elements e_i of P for which $\mathbb{E}e_i = e_i$ shall be referred to as the stationary distinguished elements of P with respect to $\mathbb{E}$. These are identical with the distinguished elements of $\mathbb{E}P$.

Theorem 2.5. For all $a \in P$ and all i , $d_i(\mathbb{E}a) \geq \mathbb{E}d_i(a)$.

Proof. $\mathbb{E}a \geq a$ implies that for all $0 \leq i \leq n-1$, $d_i(\mathbb{E}a) \geq d_i(a)$ and with $\mathbb{E}$ being a closure operator $\mathbb{E}d_i(\mathbb{E}a) \geq \mathbb{E}d_i(a)$ results. Since $\mathbb{E}B$ is the base algebra of $\mathbb{E}P$, $d_i(\mathbb{E}a) \in \mathbb{E}B$ and it follows that $\mathbb{E}d_i(\mathbb{E}a) \stackrel{=}{=} d_i(\mathbb{E}a)$ giving $d_i(\mathbb{E}a) \geq \mathbb{E}d_i(a)$ for all $a \in P$.

Theorem 2.6. $\mathbb{E}d_i(a) \geq d_i(\mathbb{E}a)$ for all a and i iff $\mathbb{E}e_i = e_i$ for all i .

Proof. Assume that $\mathbb{E}d_i(a) \geq d_i(\mathbb{E}a)$ for all a and i . Since $\mathbb{E}a \geq a$ gives $d_i(\mathbb{E}a) \geq d_i(a)$, the inequality $\mathbb{E}d_i(a) \geq d_i(\mathbb{E}a) \geq d_i(a)$ results. Now let $a = e_j$ which gives $\mathbb{E}d_i(e_j) \geq d_i(\mathbb{E}e_j) \geq d_i(e_j)$ for all $1 \leq i \leq n-1$. For $i \leq j$, $d_i(e_j) = 1$ and by the above inequality, $d_i(\mathbb{E}e_j) = 1$. For $i > j$, $d_i(e_j) = 0$ so by the above

inequality and $\exists d_i(e_j) = \exists 0 = 0$, $d_i(\exists e_j) = 0$. From these facts it

follows that $\exists e_j = \bigcup_{i=1}^{n-1} d_i(\exists e_j)e_i = e_j$.

Assume now that $\exists e_i = e_i$ for all i . One has then the following:

$$\exists a = \exists \bigcup_{i=1}^{n-1} d_i(a)e_i = \bigcup_{i=1}^{n-1} \exists(d_i(a)e_i) \leq \bigcup_{i=1}^{n-1} \exists d_i(a)\exists e_i = \bigcup_{i=1}^{n-1} \exists d_i(a)e_i \text{ which}$$

implies that $d_i(\exists a) \leq \exists d_i(a)$ for all a and i .

The two above theorems show that $d_i(\exists a) = \exists d_i(a)$ for all a and i iff $\exists e_i = e_i$ for all i or otherwise expressed that

$$\exists a = \bigcup_{i=1}^{n-1} \exists d_i(a)e_i \text{ for all } a \text{ iff } \exists e_i = e_i \text{ for all } i.$$

Definition 2.7. $\exists$ is a natural existential quantifier iff $\exists$ is an existential quantifier and for all $a \in P$,

$$\exists a = \bigcup_{i=1}^{n-1} \exists d_i(a)e_i.$$

It shall now be demonstrated that each Post existential quantifier can be constructed as in theorem 2.3 from the Boolean existential quantifier that it determines. Firstly notice that a natural ordering can be set up between the existential quantifiers on the Post algebra P as follows: $\exists_1 \leq \exists_2$ iff for all $a \in P$, $\exists_1(a) \leq \exists_2(a)$. It should be obvious that the relation will be a partial ordering.

Theorem 2.8. Let $P = \langle e_0, \dots, e_{n-1}; B \rangle$ be a Post algebra and $\exists$ a Post quantifier on P where $\exists P = \langle e_{k_0}, \dots, e_{k_m}; \exists B \rangle$ then for all $a \in P$

$$\exists a = \bigcup_{j=1}^m \exists d_{k_{j-1}+1}(a) e_{k_j}.$$

Proof. Since $\exists$ restricted to B is a Boolean existential quantifier

the definition of $\exists^*$ by $\exists^* a = \bigcup_{j=1}^m \exists d_{k_{j-1}+1}(a) e_{k_j}$ for all $a \in P$ has

a sense by theorem 2.3. For $k_{j-1} < i \leq k_j$, $\exists(d_i(a) e_i) \leq \exists d_i(a) \exists e_i =$

$= \exists d_i(a) e_{k_j} \leq \exists d_{k_{j-1}+1}(a) e_{k_j}$ which implies that

$$\exists a \leq \bigcup_{j=1}^m \exists d_{k_{j-1}+1}(a) e_{k_j} = \exists^* a \text{ for all } a \in P.$$

As $\exists P = \langle e_{k_0}, \dots, e_{k_m}; \exists B \rangle$ is a Post subalgebra $\exists a$ can be expressed

by theorem 1.10 as $\bigcup_{j=1}^m d_{k_{j-1}+1}(\exists a) e_{k_j}$ for all $a \in P$. Hence for

$$a \in P, \exists a = \bigcup_{j=1}^m d_{k_{j-1}+1}(\exists a) e_{k_j} \geq \bigcup_{j=1}^m \exists d_{k_{j-1}+1}(a) e_{k_j} \quad (\text{by theorem 2.5})$$

$$= \exists^* a.$$

The proof of theorem 2.3 and theorem 2.8 shows that given a Post algebra $\langle e_0, \dots, e_{n-1}; B \rangle$ every Boolean existential quantifier on B can be uniquely extended to a Post existential quantifier by choosing a set $0 = e_{k_0} < e_{k_1} < \dots < e_{k_m} = 1$ as the stationary distinguished elements and each Post quantifier can be determined as a mapping of this sort.

In what follows let P be a Post algebra and let $\mathbb{E}$ be a fixed Boolean quantifier on B . Each sequence $0 = k_0 < k_1 < \dots < k_m = n-1$ will determine with $\mathbb{E}$ a unique Post existential quantifier.

Theorem 2.9. Let $0 = k_0 < k_1 < \dots < k_m = n-1$ determine $\mathbb{E}$ and $0 = k'_0 < k'_1 < \dots < k'_\ell = n-1$ determine $\mathbb{E}'$, then $\mathbb{E} \leq \mathbb{E}'$ iff $\{k'_j : j = 0, \dots, \ell\} \subset \{k_j : j = 0, \dots, m\}$.

Proof. Let $\mathbb{E} \leq \mathbb{E}'$. One then has $\mathbb{E}e_{k'_j} \leq \mathbb{E}'e_{k'_j}$ which gives

$\mathbb{E}e_{k'_j} \leq e_{k'_j}$. So $\mathbb{E}e_{k'_j} = e_{k'_j}$ which means that $e_{k'_j}$ is a stationary

element for $\mathbb{E}$ giving $\{k'_j : j = 0, \dots, \ell\} \subset \{k_j : j = 0, \dots, m\}$.

Let $\{k'_j : j = 0, \dots, \ell\} \subset \{k_j : j = 0, \dots, m\}$ and define $k'_j = k_{p_j}$ for

$j = 0, \dots, \ell$. Therefore $\mathbb{E}a = \bigcup_{i=1}^m \mathbb{E}d_{k_{i-1}+1}(a)e_{k_i} = \bigcup_{j=1}^{\ell} \bigcup_{i=p_{j-1}+1}^{p_j} \mathbb{E}d_{k_{i-1}+1}(a)e_{k_i}$

$\mathbb{E}d_{k_{i-1}+1}(a)e_{k_i} \leq \bigcup_{j=1}^{\ell} \bigcup_{i=p_{j-1}+1}^{p_j} \mathbb{E}d_{k_{i-1}+1}(a)e_{k_{p_j}} =$

$= \bigcup_{j=1}^{\ell} \mathbb{E}d_{k_{p_{j-1}}+1}(a)e_{k_{p_j}} = \bigcup_{j=1}^{\ell} \mathbb{E}d_{k'_{j-1}+1}(a)e_{k'_j} = \mathbb{E}'a$ for all $a \in P$

and it follows that $\mathbb{E} \leq \mathbb{E}'$.

In the theory of the monadic Boolean algebras a dual existential quantifier is called a universal quantifier. With this concept it is then possible to demonstrate that if $\exists$ is an existential quantifier on a Boolean algebra B with the complement operator of B , then $-\exists-$ is a universal quantifier on B and every universal quantifier on B can be obtained in this way. The definition of the Post universal quantifier which includes the definition of the Boolean universal quantifier will therefore be given in the following form. Let P be a Post algebra.

Definition 2.10. An operator $V:P \longrightarrow P$ is a Post universal quantifier iff

- (1) V is an interior operator, and
- (2) VP is a Post subalgebra.

Using the facts that a Post algebra P has a dual automorphism [5] and that, the concepts of a Post subalgebra and of a complemented element of P , are self-dual, a Post universal quantifier could also be defined as a dual Post existential quantifier of P . Using the duals of theorems 2.2, 2.4, 2.5, 2.6 and 2.9 we can demonstrate that a Post universal quantifier is uniquely determined by the Boolean universal quantifier that it determines and by its stationary distinguished elements. Also any universal Boolean quantifier on the base algebra of a Post algebra can be extended to an universal Post quantifier. Indeed, if $P = \langle e_0, \dots, e_{n-1} : B \rangle$ is a Post algebra and V is a Post universal quantifier on P where $VP = \langle e_{k_0}, \dots, e_{k_m} : VB \rangle$ then for all $a \in P$, $Va =$

$$= \bigcup_{j=1}^m \bigvee_{k_j} (a) e_{k_j}.$$

Since to each Boolean existential quantifier $\exists$ there is associated a unique Boolean universal quantifier $-\exists-$ it is natural to define the Post universal quantifier associated with an Post existential quantifier $\exists$ on a Post algebra as the Post universal quantifier determined by the Boolean universal quantifier associated with the Boolean existential quantifier arising by restricting $\exists$ to B and the stationary distinguished elements of $\exists$. Let us now define the concept of a monadic Post algebra.

Definition 2.11.

- I. $(P, \exists)$ is a monadic Post algebra iff
- (1) P is a Post algebra, and
 - (2) $\exists$ is a Post existential quantifier defined on P .
- II. A monadic Post algebra $(P, \exists)$ is natural iff $\exists$ is natural.

2. Functional Monadic Post Algebras.

In the development of the monadic Boolean algebras, Halmos [6] demonstrated that every such algebra is isomorphic to a functional monadic Boolean algebra. Because of the importance that the functional monadic algebra possesses in the Boolean case, it is natural to define such a concept for the Post case. However, before this can be accomplished the following theorem must be demonstrated after which the definition will be given.

Theorem 2.12. Let $P = \langle e_0, \dots, e_{n-1} : B \rangle$ be a Post algebra and X a non-void set then P^X will be a Post algebra of the same order.

Proof. The lattice structure of P^X is induced in the usual fashion: if $p, q \in P^X$ then $(p \vee q), (p \wedge q) \in P^X$ where for all $x \in X$, $(p \vee q)(x) = p(x) \vee q(x)$ and $(p \wedge q)(x) = p(x) \wedge q(x)$.

Since P is a bounded distributive lattice P^X will also be a bounded distributive lattice where the bounded elements, 0 and 1 , of P^X are defined by the equations $1(x) = 1$ and $0(x) = 0$ for all $x \in X$.

The first condition of def. 1.1 will now be shown to hold. The set B^X being the set of complemented elements of P^X implies that B^X is a sub lattice of P^X and B^X is a Boolean algebra. Let $E_i (i = 0, \dots, n-1)$ be the elements of P^X defined by requiring that $E_i(x) = e_i$ for $i = 0, \dots, n-1, x \in X$. Define functions $D_i : P^X \rightarrow B^X$ ($i = 1, \dots, n-1$) as follows: for all $p \in P^X$ and all $x \in X$, $D_i(p)(x) = d_i(p(x))$. Notice that $p(x) \in P$ for all $x \in X$ implies

that $p(x) = \bigcup_{i=1}^{n-1} d_i(p(x))e_i$. Consider now the function $\bigcup_{i=1}^{n-1} D_i(p)E_i$.

For all $x \in X$, $(\bigcup_{i=1}^{n-1} D_i(p)E_i)(x) = \bigcup_{i=1}^{n-1} D_i(p)(x)E_i(x) = \bigcup_{i=1}^{n-1} d_i(p(x))e_i =$
 $= p(x)$ which means that $p = \bigcup_{i=1}^{n-1} D_i(p)E_i$. Directly from the definition

of the functions E_i and D_i one has that $0 = E_0 \leq E_1 \leq \dots \leq E_{n-1} = 1$ and that $D_j(p) \geq D_i(p)$ when $i \geq j$. Therefore the first condition of def. 1.1 is satisfied and indeed we have found a monotonic representation for the functions of P^X .

The second condition of def. 1.1 will now be shown to hold. Let $F \in B^X$ and let $F E_i \leq E_{i-1}$ for some i . It then follows that for all $x \in X$, $(F E_i)(x) \leq E_{i-1}(x)$ or that $F(x)e_i \leq e_{i-1}$ which implies, since P is a Post algebra, that for all $x \in X$, $F(x) = 0$ or that $F = 0$.

Definition 2.13.

(A, E) is a P -valued functional monadic Post algebra with domain $X \neq \emptyset$ iff

- (1) A is a Post subalgebra of P^X ,
- (2) for all $p \in A$, $\bigcup_{x \in X} p(x)$ exists, and

- (3) A is closed with respect to the operator E defined by

$$(Ep)(y) = \bigcup_{x \in X} p(x) \text{ for all } p \in A \text{ and } y \in X.$$

Theorem 2.14. If (A, E) is a P -valued functional monadic Post algebra with domain $X \neq \emptyset$ then $\bigcap_{x \in X} p(x)$ exists for all $p \in A$, A

is closed with respect to the operator V defined by

$$(Vp)(y) = \bigcap_{x \in X} p(x) \text{ for } y \in X, \quad (A, E) \text{ is a natural monadic Post algebra,}$$

and V is the universal Post quantifier associated with the existential Post quantifier.

Proof. Firstly, notice that A being a Post subalgebra of P^X implies that there exist functions E_{k_i} ($i = 0, \dots, m$) and a Boolean subalgebra $\hat{A}$ of B^X such that $A = \langle E_{k_0}, \dots, E_{k_m} : \hat{A} \rangle$.

If $p \in \hat{A}$, then $p \in A$ and from def. 2.13 $\bigcup_{x \in X} p(x)$ and $\exists p$ exist. As $p \in B^X$, $\bigcup_{x \in X} p(x)$ will belong to B by theorem 1.5 which implies that $\exists p \in B^X$. By hypothesis, $\exists p \in A$ giving $\exists p \in \hat{A}$ which results in $\hat{A}$ being closed with respect to the operator $\exists$. $\hat{A}$ being a Boolean algebra closed with respect to $\exists$ results in $p \in \hat{A}$ giving $\neg p \in \hat{A}$ which results in $\exists(\neg p) \in \hat{A}$ and $\neg \exists(\neg p) \in \hat{A}$. Considering $\neg \exists(\neg p) \in \hat{A}$ one finds that $\bigcup_{x \in X} (\neg p(x))$ exists and belongs to B or, otherwise, that $\bigcap_{x \in X} p(x)$ exists and belongs to B . Hence $(\neg \exists(\neg p))(y) = \bigcap_{x \in X} p(x)$.

If $p \in A$, then $D_{k_{j-1}+1}(p) \in \hat{A}$ for $j = 1, \dots, m$ which, by the above, results in $\neg \exists(\neg D_{k_{j-1}+1}(p)) \in \hat{A}$ for $j = 1, \dots, m$. Let V' be

a mapping of A defined by requiring that $V'p = \bigcup_{j=1}^m (\neg \exists(\neg D_{k_{j-1}+1}(p)))^{E_{k_j}}$

for every $p \in A$. A is then closed with respect to V' and

$\bigcap_{x \in X} (D_{k_{j-1}+1}(p))(x)$ exists for every $p \in A$. Therefore

$\bigcap_{x \in X} d_{k_{j-1}+1}(p(x))$ exists for every $p \in A$ and

$$(\forall' p)(y) = \bigcup_{j=1}^m \bigcap_{x \in X} d_{k_{j-1}+1}(p(x)) e_{k_j} \quad \text{for } p \in A \text{ and } y \in X. \quad \text{For } p \in A,$$

$$k_{j-1} < i \leq k_j, \quad j = 1, \dots, m, \quad \text{let } D_i(p) = D_{k_{j-1}+1}(p). \quad \text{Then } \bigcap_{x \in X} d_i(p(x))$$

exists and equals $\bigcap_{x \in X} d_{k_{j-1}+1}(p(x))$. Hence $\bigcap_{x \in X} p(x)$ exists for every

$$p \in A, \quad \forall \text{ exists, and } (\forall' p)(y) = \bigcup_{i=1}^{n-1} \bigcap_{x \in X} d_i(p(x)) e_i = \bigcap_{x \in X} p(x) = (\forall p)(y)$$

for $p \in A, y \in X$. A is, therefore, closed with respect to $\forall$. Note

that it has also been shown from the above that $\hat{A}$ is closed with respect

to $\forall$ and that for all $p \in A, \forall p = \bigcup_{j=1}^m D_{k_j}(p) E_{k_j}$.

By the above ~~the set~~ $(\hat{A}, \hat{\mathcal{E}})$, where $\hat{\mathcal{E}}(\mathcal{J})$ is the restriction of $\mathcal{E}(\mathcal{J})$ to $\hat{A}$, is a functional monadic Boolean algebra which as was shown by Halmos [6], is a monadic Boolean algebra and $\hat{\forall}$ is the associated universal Boolean quantifier. By the former work $\hat{\mathcal{E}}$ and $\hat{\forall}$ can be extended to quantifiers on A say to $\mathcal{E}^*$ and $\forall^*$ with the stationary distinguished elements E_k ($j = 0, \dots, m$). It will now be demonstrated that $\mathcal{E}^* = \mathcal{E}$ and $\forall^* = \forall$. ~~on A~~ . If $p \in A$ and

$$A = \langle E_{k_0}, \dots, E_{k_m} : \hat{A} \rangle \quad \text{then } p = \bigcup_{j=1}^m D_{k_{j-1}+1}(p) E_{k_j}. \quad \text{For all } p \in A,$$

$$\mathcal{E}(p)(y) = \bigcup_{x \in X} p(x) = \bigcup_{x \in X} \left(\bigcup_{j=1}^m d_{k_{j-1}+1}(p(x)) e_{k_j} \right) = \bigcup_{j=1}^m \left(\bigcup_{x \in X} d_{k_{j-1}+1}(p(x)) \right) e_{k_j}$$

$$(\text{by theorem 1.4}) = \bigcup_{j=1}^m \left(\bigcup_{y \in X} D_{k_{j-1}+1}(p)(y) \right) E_{k_j}(y) = \bigcup_{j=1}^m (\mathcal{E} D_{k_{j-1}+1}(p))(y) E_{k_j}(y) =$$

$$= \left(\bigcup_{j=1}^m \mathcal{E} D_{k_{j-1}+1}(p) E_{k_j} \right)(y) \quad \text{for all } y \in X. \quad \text{Therefore for all } p \in A,$$

$\exists p = \bigcup_{j=1}^m \exists D_{k_{j-1}+1}(p)E_{k_j}$ which means that $\exists p = \exists * p$ for all $p \in A$

because by definition $\exists * p = \bigcup_{j=1}^m \exists D_{k_{j-1}+1}(p)E_{k_j}$. Since it is known

that $\forall p = \bigcup_{j=1}^m \forall D_{k_j}(p)E_{k_j}$ and by definition that $\forall * p = \bigcup_{j=1}^m \forall D_{k_j}(p)E_{k_j}$,

$\forall * p = \forall p$ for all $p \in A$. As $\forall^*$ is the universal Post quantifier on A associated with $\exists^*$ the proof is completed.

3. Constants on a monadic Post algebra.

In the theory of monadic Boolean algebras there occurs a concept known as a constant for a given monadic algebra. A constant of $(B, \exists)$ is a Boolean homomorphism c of B into B satisfying the conditions $c\exists = \exists$ and $\exists c = c$. If instead of the quantifier $\exists$ the associated quantifier $\forall$ was used then it can easily be shown that the resulting definition is equivalent to the original. Therefore in generalizing the concept to Post algebras, it is desirable that the resulting concept should be independent of which of the two quantifiers is used in the definition. For the discussion on constants that is to follow let $(P, \exists)$ be a fixed monadic Post algebra and let $\forall^*$ be the Post universal quantifier on P associated with $\exists$. Let us begin by making two preliminary definitions.

I c is an existential constant on $(P, \exists)$ iff c is a Post homomorphism of P into P , $\exists c = c$ and $c\exists = \exists$.

II c is an universal constant on $(P, \exists)$ iff c is a Post homomorphism of P into P , $\forall^* c = c$ and $c\forall^* = \forall^*$.

Lemma. $\forall P = \exists P$.

Proof. Let e_{k_j} ($j = 0, \dots, m$) be the distinguished elements of $\exists P$.

Then by a result obtained in connection with theorem 2.2, $\exists P =$

$= \langle e_{k_0}, \dots, e_{k_m} : \exists B \rangle$. By a statement made after definition 2.10

$\forall P = \langle e_{k_0}, \dots, e_{k_m} : \forall B \rangle$. Since $\forall B = \exists B$ can easily be shown to be

valid it follows that $\forall P = \exists P$.

Lemma. If c is a universal constant or an existential constant then $cP = \exists P = \forall P$.

Proof. Consider the existential case on $(P, \exists)$ as the universal follows in a similar fashion. Now $c\exists = \exists$ implies $c\exists P = \exists P$ which gives that $\exists P \subset cP$ and $\exists c = c$ implies $\exists cP = cP$ which gives that $cP \subset \exists P$.

Theorem 2.15. Every existential constant on $(P, \exists)$ is a universal constant on $(P, \exists)$ and vice versa.

Proof. Assume that c is an existential constant on $(P, \exists)$. Let $a \in P$. By the last lemma $c(a) \in \forall P$. Let b be an element of P such that $c(a) = \forall b$ which results in $\forall c(a) = \forall b = \forall b = c(a)$. Therefore $\forall c = c$.

Let $a \in P$. By the last lemma $\forall(a) \in \mathbb{E}P$. Let b be an element of P such that $\forall(a) = \mathbb{E}b$ which results in $c\forall(a) = c\mathbb{E}(b) = \mathbb{E}(b) = \forall(a)$. Therefore $c\forall = \forall$. The proof in the other direction is as above.

Definition 2.16. c is a constant of $(P, \mathbb{E})$ iff

- (1) c is a Post homomorphism of P into P .
- (2) $c\mathbb{E} = \mathbb{E}$, and
- (3) $\mathbb{E}c = c$.

Notice that if c is a constant for $(P, \mathbb{E})$ then c restricted to B will be a constant for $(B, \hat{\mathbb{E}})$ where $\hat{\mathbb{E}}(\mathcal{J})$ is $\mathbb{E}(\mathcal{J})$ restricted to B .

Theorem 2.17. Let c be a constant for $(P, \mathbb{E})$ where $P = \langle e_0, \dots, e_{n-1} : B \rangle$

and $\mathbb{E}P = \langle e_{k_0}, \dots, e_{k_m} : \mathbb{E}B \rangle$ then there exist integers $\eta_j (j = 1, \dots, m)$

such that $k_{j-1} < \eta_j \leq k_j$ for $j = 1, \dots, m$ and $c(a) = \bigcup_{j=1}^m c d_{\eta_j}(a) e_{k_j}$

for all $a \in P$.

Proof. By $c\mathbb{E} = \mathbb{E}$ one has that for $j = 1, \dots, m$ $c\mathbb{E}e_{k_j} = \mathbb{E}e_{k_j}$ or

that $ce_{k_j} = e_{k_j}$. Now given an integer a such that $k_{j-1} < a < k_j$

it follows that $e_{k_{j-1}} = ce_{k_{j-1}} \leq ce_a \leq ce_{k_j} = e_{k_j}$, since c is a

lattice homomorphism, and that ce_a will be a distinguished element

say e_b of P , since c is a Post homomorphism. Using $\mathbb{E}c = c$

one has $e_b = ce_a = \mathbb{E}ce_a = \mathbb{E}e_b$ which results in $ce_a = e_{k_{j-1}}$ or e_{k_j}

and $ce_a \neq e_a$. By the above the stationary distinguished elements of c will therefore be the same as for $\mathfrak{A}$. Now for each $j = 1, \dots, m-1$, let η_j be the smallest integer i such that $k_{j-1} < i \leq k_j$ and

$$ce_i = e_{k_j}. \quad \text{Then for all } a \in P, \quad ca = c\left(\bigcup_{i=1}^{n-1} d_i(a)e_i\right) = \bigcup_{i=1}^{n-1} c(d_i(a))ce_i =$$

$$= \bigcup_{j=1}^m c(d_{\eta_j}(a))e_{k_j}. \quad \text{The last equality follows because of the definition}$$

of the η_j and because the $d_i(a)$ are monotonic decreasing.

Theorem 2.18. Let $P = \langle e_0, \dots, e_{n-1} : B \rangle$ be a Post algebra and c be a constant on $(B, \hat{\mathfrak{A}})$ then there exists a constant c^* on $(P, \mathfrak{A})$ which is an extension of c .

Proof. Let $\eta_j (j = 1, \dots, m)$ be arbitrary integers such that

$k_{j-1} < \eta_j \leq k_j (j = 1, \dots, m)$. Define an operation c^* on P as follows:

$$\text{for all } a \in P, \quad c^*(a) = \bigcup_{j=1}^m c(d_{\eta_j}(a))e_{k_j}. \quad \text{If } a \in B \text{ then } d_{\eta_j}(a) = a$$

and it follows that $c^*(a) = c(a)$ for $a \in B$ or that c^* is an extension of c . Therefore c restricted to B is a Boolean homomorphism of B into B . It is also easily shown by direct computation that c^* preserves the lattice meets and joins and that the distinguished element of P are mapped into distinguished elements of P . Therefore c^* is a Post homomorphism. Now for all $a \in P$,

$$\mathbb{E}c^*(a) = \mathbb{E}\left(\bigcup_{j=1}^m c(d_{\eta_j}(a))e_{k_j}\right) = \bigcup_{j=1}^m \mathbb{E}(cd_{\eta_j}(a))e_{k_j} = c^*(a) \quad \text{and} \quad c^*\mathbb{E}(a) =$$

$$= c^*\left(\bigcup_{j=1}^m \mathbb{E}d_{k_{j-1}+1}(a)e_{k_j}\right) = \bigcup_{j=1}^m c\mathbb{E}d_{k_{j-1}+1}(a)e_{k_j} = \mathbb{E}(a). \quad \text{Therefore from}$$

the above c^* as constructed is a constant on $(P, \mathbb{E})$.

4. Basically Monadic Simple Monadic Post Algebras.

The final considerations to be discussed in this chapter will be related to the simplicity of the monadic Post algebras. For Post algebras the concept of a lattice ideal is used in the definition of simplicity, but for the case of a monadic Post algebra a new concept of a monadic ideal must be introduced. A monadic ideal of $(P, \mathbb{E})$ is a lattice ideal of P which is closed with respect to the operator $\mathbb{E}$. One can then define $(P, \mathbb{E})$ to be monadic simple iff it has no non-^{proper} trivial monadic ideal.

Definition 2.19. An existential Post quantifier $\mathbb{E}$ defined on a Post algebra P is simple iff for all $a \in P - \{0\}$, $\mathbb{E}a = 1$.

Theorem 2.20. (Halmos [6]).

$(P, \mathbb{E})$ is monadic simple iff $\mathbb{E}$ is simple.

Proof. Let $(P, \mathbb{E})$ be monadic simple and let $a \in P$ where $a \neq 0$. Since $I = \{x : x \leq \mathbb{E}a\}$ is a non-trivial monadic ideal it follows that

$I = P$ and hence $1 \in I$ which implies that $\mathbb{E}a = 1$ whenever $a \neq 0$.

Suppose conversely that $\mathbb{E}a = 1$ whenever $a \neq 0$ and that I is a monadic ideal ^{of} $(P, \mathbb{E})$. If $a \in I$ then $\mathbb{E}a \in I$ and if $a \neq 0$ this implies that $1 \in I$ and hence $I = P$. Therefore every non-trivial ideal in P is improper.

By this result if $\mathbb{E}$ has a stationary distinguished element not equal to 0 or 1, then $(P, \mathbb{E})$ is not monadic simple. However even in this situation it is possible that $(B, \widehat{\mathbb{E}})$ is monadic simple, where $\widehat{\mathbb{E}}(\mathbb{J})$ is the restriction of $\mathbb{E}(\mathbb{J})$ to B .

Definition 2.21.

$(P, \mathbb{E})$ is basically monadic simple iff $(B, \widehat{\mathbb{E}})$ is monadic simple.

By the definition of a simple monadic Post algebra, it is known that such an algebra has no proper non-trivial monadic ideals. What is the situation for the basically monadic simple algebras? Notice that by a monadic prime ideal of $(P, \mathbb{E})$ is meant a monadic ideal of $(P, \mathbb{E})$ which is a prime ideal of P .

Theorem 2.22. If $(P, \mathbb{E})$ is basically monadic simple but not monadic simple then the base algebra is $\{0, 1\}$ iff P has a non-trivial monadic prime ideal. (For the rest of chapter let $2 = \{0, 1\}$.)

Proof. It is known that $\mathbb{E}a = \bigcup_{j=1}^m \mathbb{E}d_{k_{j-1}+1}(a)e_{k_j}$ if $e_{k_j} (j = 0, \dots, m)$

~~distinguished distinguished~~ are the stationary elements of $\mathbb{E}$. If $0, 1$ were the only stationary ~~distinguished~~

elements then $0 = e_{k_0} < e_{k_1} = 1$ and $\mathbb{E}a = \bigcup_{j=1}^1 \mathbb{E}d_{k_{j-1}+1}(a)e_{k_j} =$

$= \mathbb{E}d_1(a) = \widehat{\mathbb{E}d}_1(a)$. Since $\widehat{\mathbb{E}}$ is simple it follows that $\mathbb{E}$ is simple

contrary to the hypothesis of this theorem. There exists, therefore,

an $e_\alpha \neq 0, 1$ such that $\mathbb{E}e_\alpha = e_\alpha$. Consider the set $\{x : x \leq e_\alpha\}$. When

$B = 2$, P reduces to the chain of its distinguished elements from which

the set $\{x : x \leq e_\alpha\}$ is a non-trivial monadic prime ideal.

Assume now that A is a non-trivial monadic prime ideal. By the work of ~~Tracyk~~ ^{Trackzyk} [13] $N = A \cap B$ will form a prime ideal in B .

Since A and B are closed with respect to $\mathbb{E}$, N will also be closed with respect to $\mathbb{E}$, and as B is monadic simple $\{0\}$ will be the only proper monadic ideal. Using the property of a Boolean algebra that an ideal is prime iff it is maximal, it follows that $\{0\}$ is prime in B iff $B = 2$. Since $N = \{0\}$ is prime, $B = 2$.

Corollary. If $(P, \mathbb{E})$ is basically monadic simple but not monadic simple then $B \neq 2$ iff P has no non-trivial monadic prime ideals.

For a further discussion on this question consider the section on polyadic Post ideals in chapter 4. The next two theorems will result in a representation for the basically monadic simple monadic Post algebras.

Theorem 2.23. $(P, \exists)$ is basically monadic simple iff $\exists P \subset \{e_i : i = 0, \dots, n-1\}$ where $e_i (i = 0, \dots, n-1)$ are the distinguished elements of P .

Proof. $(P, \exists)$ being basically monadic simple means that $(B, \hat{\exists})$ is monadic simple or that $\hat{\exists}$ is simple. Now for all $a \in P$,

$$\exists a = \bigcup_{j=1}^m \exists d_{k_{j-1}+1}^{(a)} e_{k_j} \text{ for some } e_{k_j} (j = 0, \dots, m). \quad \hat{\exists} \text{ being simple}$$

implies that $\exists a = e_{k_j}$ for some j .

Assume that $\exists P \subset \{e_i : i = 0, \dots, n-1\}$. If $a \neq 0$ then $0 < a \leq \exists a$, and as $\exists B \subset B$ and $0, 1$ are the only distinguished elements belonging to B , it follows that for $a \in B$ and $a \neq 0$ that $\exists a = 1$. B is therefore monadic simple.

Theorem 2.24. If $(P, \exists)$ is natural and $P = \langle e_0, \dots, e_{n-1} : B \rangle$ then $(P, \exists)$ is basically monadic simple iff $(P, \exists)$ is isomorphic to a functional monadic Post algebra whose elements form a Post subalgebra of $\{e_i : i = 0, \dots, n-1\}^X$ where X is a non-void set.

Proof. As $\{e_i : i = 0, \dots, n-1\}$ is a complete Post algebra it is immediate that $\{e_i : i = 0, \dots, n-1\}^X$ can be considered as a functional monadic Post algebra with $\exists$ being the existential quantifier defined for def. 2.13. For each $p \in \{e_i : i = 0, \dots, n-1\}^X$ there exists an integer

k such that $(\exists p)(y) = \bigcup_{x \in X} p(x) = e_k$ for all $y \in X$ which means that

$\exists p = E_k$. From theorem 2.23 this results in the algebra being basically monadic simple. Since any monadic Post subalgebra of a basically monadic simple algebra is basically monadic simple, an algebra isomorphic

to a monadic Post subalgebra of $(\{e_i : i = 0, \dots, n-1\}^X, \exists)$ will be basically monadic simple.

Assume now that $(P, \exists)$ is basically monadic simple. Let h be an isomorphism of B onto a Boolean subalgebra B' of 2^X . The existence of such a function follows from Stone's representation theorem [12] for Boolean algebras. 2^X can be considered as a functional monadic Boolean algebra with the existential quantifier $\exists'_f$. Let $\exists'$ be the mapping $\exists'_f$ restricted to B' . Then for $p \in B'$, $\exists'p = E_0$ if $p = E_0$ and $\exists'p = E_{n-1}$ if $p \neq E_0$, resulting in $\exists'$ being a Boolean existential quantifier on B' . The equality $h(\exists a) = \exists'(ha)$ holds since either side of this equation equals E_0 if $a = 0$ and E_{n-1} if $a \neq 0$. Hence h is even a monadic isomorphism of $(B, \hat{\exists})$ onto $(B', \exists')$. Extend the mapping h to a mapping $f: P \rightarrow \langle E_0, \dots, E_{n-1} : 2^X \rangle$ as follows:

$$f(a) = \bigcup_{i=1}^{n-1} h(d_i(a))E_i \quad \text{for all } a \in P. \quad \bigcup_{i=1}^{n-1} h(d_i(a))E_i \text{ is, then, easily seen}$$

to be the monotonic representation of $f(a)$ in the algebra $\langle E_0, \dots, E_{n-1} : 2^X \rangle$, and f will be an extension of h since for all

$$a \in B, \quad f(a) = \bigcup_{i=1}^{n-1} h(d_i(a))E_i = \bigcup_{i=1}^{n-1} h(a)E_i = h(a). \quad \text{Using this fact and the}$$

fact that $f(e_i) = E_i$ which follows from the construction of f , f is seen to be a Post isomorphism of P onto $f(P)$. This includes that the image $f(P)$ is a Post subalgebra of the algebra

$\langle E_0, \dots, E_{n-1} : 2^X \rangle$. Let $\exists_f$ be the existential quantifier associated

with the functional monadic Post algebra $\langle E_0, \dots, E_{n-1}; 2^X \rangle$. Looking next at the behaviour of f with the quantifiers we see that

$$f(\mathbb{E}a) = \bigcup_{i=1}^{n-1} h(d_i(\mathbb{E}a))E_i = \bigcup_{i=1}^{n-1} h(\mathbb{E}d_i(a))E_i \quad (\text{since } \mathbb{E} \text{ is natural;}$$

$$\text{see theorems 2.5 and 2.6}) = \bigcup_{i=1}^{n-1} \mathbb{E}'h(d_i(a))E_i \quad (\text{since } h \text{ is a monadic}$$

$$\text{isomorphism of } (B, \hat{\mathbb{E}}) \text{ onto } (B', \mathbb{E}')) = \mathbb{E}_f \bigcup_{i=1}^{n-1} h(d_i(a))E_i \quad (\text{since } \mathbb{E}_f$$

is natural by theorem 2.14 and $\mathbb{E}_f a = \mathbb{E}'a$ for $a \in B'$). Hence

$f(P)$ is closed with respect to $\mathbb{E}_f$. Let $\mathbb{E}^*$ be the restriction of

$\mathbb{E}_f$ to $f(P)$, then $(f(P), \mathbb{E}^*)$ is a monadic Post algebra, and f is

a monadic isomorphism of $(P, \mathbb{E})$ onto $(f(P), \mathbb{E}^*)$ completing the proof.

CHAPTER III

POLYADIC POST ALGEBRAS

This chapter will deal with concepts which are generalizations of the concepts introduced by Paul Halmos [6] in his paper on polyadic Boolean algebras. The idea of a Post quantifier will be that of the last chapter.

1. Quantifier and Transformation Post Algebras.

Definition 3.1. $(P, I, \mathfrak{E})$ is a quantifier Post algebra iff

- (1) P is a Post algebra,
- (2) I is a set,
- (3) $\mathfrak{E}$ is a mapping of the set of the subsets of I into the set of the Post existential quantifiers on P ,
- (4) $\mathfrak{E}(\emptyset)$ is the identity on P , and
- (5) for all $J, K \subset I$, $\mathfrak{E}(J \cup K) = \mathfrak{E}(J)\mathfrak{E}(K)$.

Example. Let $P = \langle e_0, \dots, e_{n-1} : B \rangle$ be a Post algebra, $(B, I, \hat{\mathfrak{E}})$ be a quantifier Boolean algebra,

$$\{e_{k_j} : j = 0, \dots, m\} \subset \{e_{k'_j} : j = 0, \dots, l\} \subset \{e_i : i = 0, \dots, n-1\} \text{ where}$$

$$0 = e_{k_0} = e_{k'_0} \text{ and } 1 = e_{k_m} = e_{k'_l}, \text{ and } c \text{ be an element of } I.$$

Define a mapping $\mathfrak{E}$ from the set of subsets of I into the set of the Post existential quantifiers on P as follows:

(1) $\mathbb{E}(\emptyset)$ is the identity on P , and

(2) if $J \neq \emptyset$ then for all $a \in P$,

$$\mathbb{E}(J)a = \bigcup_{j=1}^{\ell} \widehat{\mathbb{E}}(J) d_{k'_{j-1}+1}(a) e_{k'_j} \quad \text{if } c \notin J, \text{ and}$$

$$\mathbb{E}(J)a = \bigcup_{j=1}^m \widehat{\mathbb{E}}(J) d_{k_{j-1}+1}(a) e_{k_j} \quad \text{if } c \in J.$$

By definition $\mathbb{E}(\emptyset)$ is the identity and by considering the various cases that result when $c \in J \cup K$ or $c \notin J \cup K$ it is easily shown that $\mathbb{E}(J \cup K) = \mathbb{E}(J) \mathbb{E}(K) = \mathbb{E}(K) \mathbb{E}(J)$.

Definition 3.2. A quantifier Post algebra $(P, I, \mathbb{E})$ is called natural iff $\mathbb{E}(J)$ is natural for all subsets J of I .

For what follows the notation $D(J)$ shall denote the set of stationary elements of P under $\mathbb{E}(J)$ for the given quantifier algebra under consideration.

Theorem 3.3. If $(P, I, \mathbb{E})$ is a quantifier Post algebra then $D(J \cup K) = D(J) \cap D(K)$.

Proof. Let $a \in P$. If $\mathbb{E}(J)a = a$ and $\mathbb{E}(K)a = a$ then $\mathbb{E}(K \cup J)a = \mathbb{E}(K)(\mathbb{E}(J)a) = \mathbb{E}(K)a = a$. If $\mathbb{E}(K \cup J)a = a$ then $\mathbb{E}(K)a = \mathbb{E}(K)(\mathbb{E}(K \cup J)a) = (\mathbb{E}(K)\mathbb{E}(K))(\mathbb{E}(J)a) = \mathbb{E}(K)\mathbb{E}(J)a = \mathbb{E}(K \cup J)a = a$, and similarly $\mathbb{E}(J)a = a$. Hence $a \in D(K \cup J)$ iff $a \in D(K) \cap D(J)$.

Corollary.

(1) $J \subset K$ implies $D(K) \subset D(J)$, and

(2) $D(I) \subset D(J) \subset D(\emptyset)$.

Definition 3.5. (P, I, S) is a transformation Post algebra iff

(1) P is a Post algebra,

(2) I is a set,

(3) S is a mapping of I^I into the Post homomorphisms of P into P .

(4) If δ is the identity map on I , then $S(\delta)$ is the identity map on P , and

(5) for all $r, \sigma \in I^I$, $S(r\sigma) = S(r)S(\sigma)$.

Example. Let $P = \langle e_0, \dots, e_{n-1} : B \rangle$ be a Post algebra, I a non-void finite set, $(E, I, \hat{S})$ a Boolean transformation algebra and

$\{e_{k_j} : j = 0, \dots, m\} \subset \{e_i : i = 0, \dots, n-1\}$ where $e_{k_0} = 0$ and $e_{k_m} = 1$.

In what follows the fact that a transformation on I is onto iff it is one to one which results in σr being onto iff σ and r are onto shall be used. Define a mapping S from I^I to the Post homomorphisms of P into P as follows:

(1) $S(\delta)$ is the identity on P , and

(2) if $r \neq \delta$ then for all $a \in P$,

$$S(r)a = \bigcup_{i=1}^{n-1} \hat{S}(r)d_i(a)e_i \quad \text{if } r \text{ is onto, and}$$

$$S(r)a = \bigcup_{j=1}^m \hat{S}(r)d_{k_{j-1}+1}(a)e_{k_j} \quad \text{if } r \text{ is not onto.}$$

It is easily demonstrated that $S(r)$ is a Post homomorphism of P into P . Since $S(\delta)$ is the identity by definition and $S(r\sigma) = S(r)S(\sigma)$ which is easily demonstrated from the fact that $r\sigma$ is onto iff r and σ are onto, it follows that (P, I, S) is a transformation algebra.

Definition 3.6. A transformation Post algebra (P, I, S) is natural iff for all $r \in I^I$, $S(r)$ is natural.

Theorem 3.7. If (P, I, S) is a transformation Post algebra then $S(r)S(\sigma) = S(\delta)$ implies that $S(r)$ and $S(\sigma)$ are natural.

Proof. Let $\bar{S}(r)$ and $\bar{S}(\sigma)$ be, respectively, $S(r)$ and $S(\sigma)$ restricted to $\{e_i : i = 0, \dots, n-1\}$. Since $S(r)S(\sigma) = S(\delta)$, $\bar{S}(r)\bar{S}(\sigma)$ will be onto which means that $\bar{S}(r)$ and $\bar{S}(\sigma)$ are onto and one to one. As $\bar{S}(r)$ and $\bar{S}(\sigma)$ are monotonic, they will be identity maps of $\{e_i : i = 0, \dots, n-1\}$. Therefore $S(r)$ and $S(\sigma)$ are natural.

Corollary. If r is one to one or onto then $S(r)$ will be natural.

2. Polyadic Post Algebras.

Definition 3.8. $(P, I, \mathbb{E}, S)$ is a polyadic Post algebra iff

- (1) $(P, I, \mathbb{E})$ is a quantifier Post algebra,
- (2) (P, I, S) is a transformation Post algebra,
- (3) $S(r)\mathbb{E}(J) = S(\sigma)\mathbb{E}(J)$ whenever $ri = \sigma i$ for $i \notin J$,
- (4) $\mathbb{E}(J)S(r) = S(r)\mathbb{E}(r^{-1}J)$ whenever r is one to one on $r^{-1}J$.

Example. Let $P = \langle e_0, \dots, e_{n-1} : B \rangle$ be a Post algebra, I be a finite non-void set, $(B, I, \hat{H}, \hat{S})$ be a Boolean polyadic algebra, and $\{e_{k_j} : j = 0, \dots, m\} \subset \{e_i : i = 0, \dots, n-1\}$ where $e_{k_0} = 0$ and $e_{k_m} = 1$.

Define S and H as follows:

- (1) for S use the construction of the example after def. 3.5, and
- (2) for H , $H(\emptyset)$ will be the identity map of P and for all $J \neq \emptyset$, $H(J)a = \bigcup_{j=1}^m \hat{H}(J)d_{k_{j-1}+1}(a)e_{k_j}$ for all $a \in P$.

It is immediate that (P, I, S) is a transformation Post algebra and (P, I, H) is a quantifier Post algebra by considering the former examples. That conditions (3) and (4) are satisfied can be shown by considering the various cases.

Definition 3.9. A polyadic Post algebra is called natural iff the quantifier Post and the transformation Post algebras involved are natural (see def. 3.2 and def. 3.4).

Another example of a polyadic Post algebra, which is of importance is that of the functional polyadic Post algebra. In the following, I and X are sets, $X \neq \emptyset$, P is a Post algebra, and A is a subset of $P^{(X^I)}$. If $r \in I^I$, r_x is the mapping of X^I into X^I defined by requiring that $(r_x(x))(i) = x(r(i))$ for $x \in X^I$, $i \in I$. If $x \in X^I$, $y \in X^I$ and $J \subset I$, let $xJ_x y$ mean that $x(i) = y(i)$ for every element $i \in I-J$, and if $p \in A$ and $x \in X^I$, let $\bigcup \{p(y) : y \in X^I, xJ_x y\}$ exist. S is to be the mapping of I^I into $(P^{(X^I)})^A$ defined by requiring that $((S(r))(p))(x) = p(r_x(x))$ for $r \in I^I$, $p \in A$, $x \in X^I$.

$\mathfrak{E}$ is the mapping of the power set of I into $(P^{(X^I)})^A$ defined by requiring that $((\mathfrak{E}(J))(p))(x) = \bigcup \{p(y) : y \in X^I, xJ_*y\}$ for $J \subset I$, $p \in A$, $x \in X^I$.

Definition 3.10. $(A, I, \mathfrak{E}, S)$ is called a functional polyadic Post algebra iff

- (1) A is a Post subalgebra of $P^{(X^I)}$,
- (2) A is closed with respect to $S(r)$ for $r \in I^I$, and
- (3) A is closed with respect to $\mathfrak{E}(J)$ for $J \subset I$.

Theorem 3.11. A functional polyadic Post algebra $(A, I, \mathfrak{E}, S)$ is a natural polyadic Post algebra.

Proof. It is desirable to show first of all that $A \cap B^{(X^I)}$, where B is the base algebra of P , is closed with respect to every $S(r)$ and every $\mathfrak{E}(J)$. Obviously for all $x \in X^I$, $(S(r)p)(x) = p(r_*(x)) \in B$ if $p \in B^{(X^I)}$ which means that $S(r)p \in B^{(X^I)}$. From the definition of a functional polyadic Post algebra $(S(r)p) \in A$ if $p \in A$ which means that $(S(r)p) \in A \cap B^{(X^I)}$ for $p \in A \cap B^{(X^I)}$. Since $p \in B^{(X^I)}$ implies that $\{p(y) : y \in X^I \text{ and } yJ_*x\} \subset B$ for all $x \in X^I$ and P being a Post algebra it follows from theorem 1.5 that $\bigcup \{p(y) : y \in X^I \text{ and } yJ_*x\} \in B$ for all $x \in X^I$ which implies that $(\mathfrak{E}(J)p)(x) \in B$ for all $x \in X^I$.

Since $(\mathfrak{E}(J)p) \in A$ for $p \in A$ we have that $\mathfrak{E}(J)p \in A \cap B^{(X^I)}$ for all $p \in A \cap B^{(X^I)}$. Let $\hat{A} = A \cap B^{(X^I)}$. Let $\hat{S}$ be the mapping of I^I into $\hat{A}$ defined by requiring that $\hat{S}(r)p = S(r)p$ for $r \in I^I$ and $p \in \hat{A}$ and $\hat{\mathfrak{E}}$ the mapping of the power set of I into $\hat{A}$ defined by requiring that $\hat{\mathfrak{E}}(J)p = \mathfrak{E}(J)p$ for $J \subset I$ and $p \in \hat{A}$. $(\hat{A}, I, \hat{\mathfrak{E}}, \hat{S})$ is then a functional polyadic Boolean algebra. By the work of P. Halmos [6] it is known that $(\hat{A}, I, \hat{\mathfrak{E}}, \hat{S})$ is a polyadic Boolean algebra.

The results and notation of theorem 2.12 will be utilized in what follows. The distinguished elements of P are $e_i (i = 0, \dots, n-1)$

and $E_i (i = 0, \dots, n-1)$ will be the distinguished elements of $P^{(X^I)}$.

As A is a Post subalgebra of $P^{(X^I)}$ let $E_{k_j} (j = 0, \dots, m)$ denote

the distinguished elements of A . Now it follows that for all $p \in A$,

$$r \in I^I, J \subset I, S(r)p = \bigcup_{j=1}^m D_{k_{j-1}+1}(S(r)p)E_{k_j} \text{ and}$$

$$\mathfrak{E}(J)p = \bigcup_{j=1}^m D_{k_{j-1}+1}(\mathfrak{E}(J)p)E_{k_j}. \text{ Now for all } x \in X^I,$$

$$(D_{k_{j-1}+1}(S(r)p))(x) = d_{k_{j-1}+1}((S(r)p)(x)) = d_{k_{j-1}+1}(p(r_*(x))) =$$

$$= (D_{k_{j-1}+1}(p))(r_*(x)) = (\hat{S}(r)D_{k_{j-1}+1}(p))(x), \text{ and}$$

$$(D_{k_{j-1}+1}(\mathfrak{E}(J)p))(x) = d_{k_{j-1}+1}((\mathfrak{E}(J)p)(x)) =$$

$$= d_{k_{j-1}+1}(\bigcup \{p(y) : y \in X^I \text{ and } yJ_*x\})$$

$$= \bigcup \left\{ d_{k_{j-1}+1}(p(y)) : y \in X^I \text{ and } yJ_*x \right\} \quad (\text{by theorem 1.4})$$

$$= \bigcup \left\{ (D_{k_{j-1}+1}(p))(y) : y \in X^I \text{ and } yJ_*x \right\} = (\hat{\mathbb{E}}(J)D_{k_{j-1}+1}(p))(x).$$

This means that for $r \in I^I$ and $p \in A$, $S(r)p =$

$$= \bigcup_{j=1}^m \hat{S}(r)D_{k_{j-1}+1}(p)E_{k_j} \quad \text{and that for } J \subset I \text{ and } p \in A, \quad \mathbb{E}(J)p =$$

$$= \bigcup_{j=1}^m \hat{\mathbb{E}}(J)D_{k_{j-1}+1}(p)E_{k_j}. \quad \text{From this and the fact that } (\hat{A}, I, \hat{\mathbb{E}}, \hat{S}) \text{ is a}$$

polyadic Boolean algebra, $(A, I, \mathbb{E}, S)$ can easily be seen to be a natural polyadic Post algebra.

It is now desirable to introduce a number of concepts which were introduced for the case of the polyadic Boolean algebra by P. Halmos [6] and A. Daigneault and D. Monk [2]. Let $(P, I, \mathbb{E}, S)$ be a polyadic Post algebra.

Definition 3.12.

(1) If $a \in P$ and $J \subset I$, then a is independent of J or $I-J$ supports a iff $\mathbb{E}(J)a = a$.

(2) $(P, I, \mathbb{E}, S)$ is of infinite degree iff I is infinite.

(3) $(P, I, \mathbb{E}, S)$ is of local degree γ iff γ is the smallest cardinal α such that every $a \in P$ has a support whose cardinality is less than α .

If $\gamma = \omega$ then following P. Halmos [6] the algebra will be said to be locally finite. In what follows a sufficient condition for

a polyadic Post algebra of infinite degree to be natural shall be given.

Denote the cardinality of a set I by $\bar{I}$.

Lemma. Let I be infinite, $J \subset I$, $\bar{J} = \bar{I}$, $r \in I^I$, and for all $i \in J$, $r(i) = i$, then there exist $Q, \psi \in I^I$ where Q is onto and ψ is one to one such that $r = Q\psi$.

Proof. Let f be a one to one mapping of I onto J . Define mappings Q and ψ as follows:

(1) $Q(f(i)) = i$ for $i \in I$ which means that $Q(j)$ is defined for $j \in J$, and $Q(i) = r(i)$ for $i \in I-J$,

(2) $\psi(i) = i$ for $i \in I-J$, and $\psi(i) = f(i)$ for $i \in J$.

Now Q is onto and ψ is one to one. If $i \in J$ then

$(Q\psi)(i) = Q(f(i)) = i = r(i)$ and if $i \in I-J$ then $(Q\psi)(i) = Q(\psi(i)) = Q(i) = r(i)$ which gives $Q\psi = r$.

Theorem 3.13. If $(P, I, \mathfrak{A}, S)$ is of infinite degree and of local degree γ where $\gamma \leq \bar{I}$ then $(P, I, \mathfrak{A}, S)$ is natural.

Proof. If $r \in I^I$ and for all $i \in J$, $r(i) = i$ where $\bar{J} = \bar{I}$ then $r = Q\psi$ where Q is onto and ψ is one to one. Since

$S(r) = S(Q\psi) = S(Q)S(\psi)$ and $S(Q), S(\psi)$ are natural by corollary of theorem 3.7, it follows that $S(r)$ is natural. For

$e_i (i = 1, \dots, n-2)$ there exist supports $K_i (i = 1, \dots, n-2)$ where $\bar{K}_i < \gamma (i = 1, \dots, n-2)$. As $\gamma \leq \bar{I}$, $\bar{K}_i < \bar{I} (i = 1, \dots, n-2)$ which means that $J_i = I - K_i (i = 1, \dots, n-2)$ must satisfy the condition that

$\bar{J}_i = \bar{I}$ ($i = 1, \dots, n-2$). Let r be an arbitrary element of I^I .

Define r_i ($i = 1, \dots, n-2$) as the mapping of I defined by

$r_i(j) = r(j)$ if $j \notin J_i$ and $r_i(j) = j$ if $j \in J_i$. From condition

(3) of def. 3.8 for $i = 1, \dots, n-2$, $S(r)e_i = S(r)\mathbb{E}(J_i)e_i =$

$= S(r_i)\mathbb{E}(J_i)e_i = S(r_i)e_i$. Since r_i ($i = 1, \dots, n-2$) is the identity

on J_i and $\bar{J}_i = \bar{I}$ it follows that $S(r_i)$ ($i = 1, \dots, n-2$) is natural,

giving $S(r)e_i = S(r_i)e_i = e_i$ for $i = 1, \dots, n-2$. Therefore if

$r \in I^I$ then $S(r)$ is natural.

It is necessary now to demonstrate that for all $J \subset I$, ~~that~~ $\mathbb{E}(J)$ is natural. Notice that in this demonstration we will use the fact

that $S(r)$ is natural for all $r \in I^I$. Assume that $J \neq I$ which

means that $I - J \neq \emptyset$. Let r be an arbitrary transformation on I

for which $r(I) \subset I - J$. In this case $r^{-1}J = \emptyset$ and r is therefore

one to one on $r^{-1}J$. Utilizing condition (4) of def. 3.8 it follows

that $\mathbb{E}(J)S(r) = S(r)\mathbb{E}(r^{-1}J) = S(r)\mathbb{E}(\emptyset) = S(r)$. So $\mathbb{E}(J)e_i =$

$= (\mathbb{E}(J)S(r))e_i$ (because $S(r)$ is natural) $= S(r)e_i = e_i$ for $i = 0, \dots, n-1$,

and $\mathbb{E}(J)$ is natural. Let $J \neq I, \emptyset$ and let $J \cup J' = I$ and $J \cap J' = \emptyset$.

By theorem 3.3 the set of the stationary distinguished elements of

$\mathbb{E}(I)$ is the intersection of the sets of the stationary distinguished

elements for $\mathbb{E}(J)$ and $\mathbb{E}(J')$. As $\mathbb{E}(J)$ and $\mathbb{E}(J')$ will be natural

it, therefore, follows that $\mathbb{E}(I)$ is natural.

Notice that the above theorem includes the important special case of a locally finite polyadic Post algebra of infinite degree. In the

example given after Def. 3.8, I is finite while the polyadic Post

algebra $(P, I, \mathbb{E}, S)$ is not natural if $\{e_{k_j} : j = 0, \dots, m\}$ is a proper

subset of $\{e_i : i = 0, \dots, n-1\}$. The situation that remains but for which an answer is not known is whether a polyadic Post algebra (P, I, E, S) of infinite degree and of local degree $\bar{I} + 1$ is necessarily natural. However this case is not important in logic [8].

Theorem 3.14. If (P, I, E, S) is a polyadic Post algebra and the transformation Post algebra (P, I, S) is natural then (P, I, E, S) is natural.

Proof. The proof is as found in the proof of theorem 3.13.

Theorem 3.15. Let (P, I, E, S) be a polyadic Post algebra. For all $r \in I^I$ the set of the stationary distinguished elements of $S(r)$ contains the stationary distinguished elements of $E(I)$.

Proof. By (3) of Def. 3.8, with $J = I$ and $\sigma = \delta$, $S(r)E(I) = E(I)$. Let $i \in \{0, \dots, n-1\}$ and $E(I)e_i = e_i$. Therefore $S(r)e_i = (S(r)E(I))e_i = E(I)e_i = e_i$ results in e_i being stationary under $S(r)$ proving the theorem.

Corollary. If (P, I, E, S) is a polyadic Post algebra and the quantifier Post algebra is natural then (P, I, E, S) is natural.

To complete the chapter the ideas of a constant and an equality introduced for the case of a polyadic Boolean algebra by P. Halmos [6], shall be defined for the polyadic Post algebra, with the main interest being their relationship with the natural algebras.

Definition 3.14b. A mapping c of the power set of I into the set of all Post homomorphisms of P into P is a constant for $(P, I, \mathfrak{E}, S)$ if it satisfies the following conditions: for all $J, K \subset I$ and $r \in I^I$,

$$(1) \quad c(\emptyset) \text{ is the identity on } P,$$

$$(2) \quad c(J \cup K) = c(J)c(K),$$

$$(3) \quad c(J)\mathfrak{E}(K) = \mathfrak{E}(K)c(J-K),$$

$$(4) \quad \mathfrak{E}(K)c(J) = c(J)\mathfrak{E}(K-J), \text{ and}$$

$$(5) \quad c(J)S(r) = S(r)c(r^{-1}J).$$

Theorem 3.17. If c is a constant for a polyadic Post algebra $(P, I, \mathfrak{E}, S)$ then for all $J \subset I$, $\mathfrak{E}(J)$ and $c(J)$ have the same stationary distinguished elements.

Proof. From the definition of a constant the following two equations $c(K)\mathfrak{E}(K) = \mathfrak{E}(K)c(\emptyset) = \mathfrak{E}(K)$ and $\mathfrak{E}(K)c(K) = c(K)\mathfrak{E}(\emptyset) = c(K)$ hold. If e_i is a stationary distinguished element of $\mathfrak{E}(K)$ then $c(K)\mathfrak{E}(K)e_i = \mathfrak{E}(K)e_i$ gives $c(K)e_i = e_i$ which results in e_i being a stationary distinguished element of $c(K)$. Similarly using the second equation if e_i is a stationary distinguished element of $c(K)$ then it will be a stationary element of $\mathfrak{E}(K)$.

Let (i/j) denote the element of I^I which maps i to j and maps every element of $I - \{i\}$ to itself.

Definition 3.18. An equality of $(P, I, \mathbb{E}, S)$ is an element d belonging to $P^{(I \times I)}$ which satisfies the conditions: for all $i, j \in I$, $r \in I^I$, $a \in P$,

$$(1) \quad d(i, i) = 1,$$

$$(2) \quad S(r)d(i, j) = d(ri, rj), \quad \text{and}$$

$$(3) \quad a \cap d(i, j) \leq S(i/j)a.$$

Theorem 3.19. If d is an equality for a natural polyadic Post algebra $(P, I, \mathbb{E}, S)$ where $P = \langle e_0, \dots, e_{n-1} : B \rangle$ then for all $i, j \in I$, $d(i, j) \in B$.

Proof. Let $E_k(i, j) = d_k(d(i, j))$ for all $i, j \in I$ and $1 \leq k \leq n-1$. It is desirable to show that E_k ($k = 1, \dots, n-1$) is an equality for $(B, I, \hat{\mathbb{E}}, \hat{S})$ where $\hat{S}(r)$ arises from $S(r)$ by restricting every $S(r)$ to the set B and similarly for $\hat{\mathbb{E}}$. Then using the fact that an equality if it exists for a polyadic Boolean algebra is unique [6] it follows that $d(i, j) \in B$ for all $i, j \in I$.

By definition $E_k \in B^{(I \times I)}$ ($k = 1, \dots, n-1$). Consider now the three conditions of the definition of an equality.

$$(1) \quad E_k(i, i) = d_k(d(i, i)) = d_k(1) = 1.$$

$$(2) \quad S(r)E_k(i, j) = S(r)d_k(d(i, j)) = d_k(S(r)d(i, j)) \quad (\text{Since } (P, I, \mathbb{E}, S) \text{ is natural}) = d_k(d(ri, rj)) = E_k(ri, rj).$$

(3) Let $a \in B$ and since d is an equality $a \wedge d(i, j) \leq S(i/j)a$, which implies that $d_k(a \wedge d(i, j)) \leq d_k(S(i/j)a)$ or that $d_k(a) \wedge d_k(d(i, j)) \leq d_k(S(i/j)a)$ for $1 \leq k \leq n-1$. Since $(P, I, \mathbb{E}, S)$ is natural $d_k(a) \wedge d_k(d(i, j)) \leq S(i/j)d_k(a)$, which implies $a \wedge d_k(d(i, j)) \leq S(i/j)a$ or that $a \wedge E_k(i, j) \leq S(i/j)a$, since $a \in B$. Therefore E_k ($k = 1, \dots, n-1$) is an

equality for $(B, I, \hat{\mathbb{E}}, \hat{S})$.

The above theorem implies that an equality for a natural algebra $(P, I, \mathbb{E}, S)$ will also be an equality for $(B, I, \hat{\mathbb{E}}, \hat{S})$. Therefore if there exist an equality for $(P, I, \mathbb{E}, S)$ there exists only one. Without any difficulty an equality for $(B, I, \hat{\mathbb{E}}, \hat{S})$ can be shown to be an equality for the natural algebra $(P, I, \mathbb{E}, S)$.

There is a very natural relationship between the Boolean and the Post polyadic algebras which are of infinite degree and of local degree γ where $\omega \leq \gamma \leq \bar{I}$. In the following assume that $\omega \leq \gamma \leq \bar{I}$ holds. By restricting the Post quantifiers and the Post homomorphisms that result in a Post polyadic algebra to the base algebra a polyadic Boolean algebra results. If the polyadic Post algebra is of local degree γ and of infinite degree, then it is obvious that the resulting polyadic Boolean algebra will be of local degree η where $\eta \leq \gamma$ and of infinite degree. Let us attempt to go in the other direction where we assume that $(B, I, \mathbb{E}, S)$ is a given polyadic Boolean algebra of infinite degree and of local degree γ . From the theory of Post algebras it is known that there exists a Post algebra $\langle e_0, \dots, e_{n-1}; B' \rangle$ of order n such that there exists an isomorphism h of B onto B' .

Consider for example the set $\left\{ d \in B^{\{1, \dots, n-1\}} : \text{for all } i, j \in \{1, \dots, n-1\}, i \leq j \text{ implies } d_i \geq d_j \right\}$. Define the operations $\mathbb{E}'$ and S' by the equations $\mathbb{E}'(J)h(a) = h(\mathbb{E}(J)a)$ and $S'(r)h(a) = h(S(r)a)$ which results in $(B', I, \mathbb{E}', S')$ being a polyadic Boolean algebra and h being a polyadic isomorphism of $(B, I, \mathbb{E}, S)$ onto $(B', I, \mathbb{E}', S')$. Extend the mappings $\mathbb{E}'(J)$ and $S'(r)$ on B' to P as follows: for all

$$J \subset I, r \in I^I, a \in P, \mathbb{E}^*(J)a = \bigcup_{i=1}^{n-1} \mathbb{E}'(J)d_i(a)e_i \quad \text{and} \quad S^*(r)a = \bigcup_{i=1}^{n-1} S'(r)d_i(a)e_i.$$

It is obvious that these extensions are Post quantifiers and homomorphisms and that $(P, I, \mathbb{E}^*, S^*)$ with the extensions $\mathbb{E}^*$ and S^* is a polyadic Post algebra. As I in $(B, I, \mathbb{E}, S)$ is infinite, the resulting algebra $(P, I, \mathbb{E}^*, S^*)$ will be of infinite degree. It takes a little more effort to demonstrate that $(P, I, \mathbb{E}^*, S^*)$ will be of local degree γ . Let $a \in P$. Now for $0 \leq i \leq n-1$, there exists a support K_i of $d_i(a)$ of cardinality less than γ . If $J_i = I - K_i$ then $\mathbb{E}'(J_i)d_i(a) = d_i(a)$ and $\overline{J_i} = \overline{I}$ for all $i = 1, \dots, n-1$. The set $I - J$ where

$$J = \bigcap_{i=1}^{n-1} J_i \text{ is of a cardinality less than } \gamma \text{ since } I - J = \bigcup_{i=1}^{n-1} (I - J_i) \text{ and}$$

$$\overline{I - J_i} < \gamma \text{ for } i = 1, \dots, n-1. \text{ Now } \mathbb{E}'(J)d_i(a) = \mathbb{E}'(J)\mathbb{E}'(J_i)d_i(a) = \\ = \mathbb{E}'(J \cup J_i)d_i(a) = \mathbb{E}'(J_i)d_i(a) = d_i(a) \text{ for } i \leq i \leq n-1$$

$$\text{gives } \mathbb{E}'(J)a = \bigcup_{i=1}^{n-1} \mathbb{E}'(J)d_i(a)e_i = \bigcup_{i=1}^{n-1} d_i(a)e_i = a. \text{ "a" being an}$$

arbitrary element of P , $(P, I, \mathbb{E}^*, S^*)$ is of local degree γ . Since a polyadic Post algebra of infinite degree and of local degree γ where $\gamma \leq \overline{I}$ is natural (theorem 3.13), the above construction gives all such algebras when $\omega \leq \gamma \leq \overline{I}$. Notice also that for these cases the local degree of $(B, I, \widehat{\mathbb{E}}, \widehat{S})$ and $(P, I, \mathbb{E}, S)$ will be equal.

CHAPTER IV

POLYADIC LOGICS AND IDEALS

The development of this chapter will be to consider firstly the polyadic ideal in a polyadic Post algebra after which the concept of a polyadic logic and of a model and interpretation for the logic shall be introduced which will lead to questions of semantic consistency and completeness.

1. Polyadic ideals.

Definition 4.1. Let $P = \langle e_0, \dots, e_{n-1} : B \rangle$

(1) Δ is a polyadic ideal of $(P, I, \mathfrak{E}, S)$ iff Δ is a lattice ideal of P and closed with respect to $\mathfrak{E}(J)$ and $S(r)$ for all $J \subset I$ and $r \in I^I$.

(2) A polyadic Post algebra is polyadic simple iff it has no non-trivial proper polyadic ideals.

(3) A polyadic Post algebra $(P, I, \widehat{\mathfrak{E}}, \widehat{S})$ is basically polyadic simple iff $(B, I, \widehat{\mathfrak{E}}, \widehat{S})$ is polyadic simple.

In the following we shall sometimes say polyadic ideal instead of polyadic ideal of $(P, I, \mathfrak{E}, S)$.

Theorem 4.2. If $(P, I, \mathfrak{E}, S)$ is a polyadic Post algebra which is not basically polyadic simple and if for $J \subset I$, $r \in I^I$,

$\mathfrak{E}(J)e_{k-1} = e_{k-1}$, $S(r)e_{k-1} = e_{k-1}$, $\mathfrak{E}(J)e_k = e_k$, $S(r)e_k = e_k$ where

$1 \leq k \leq n-2$ and $\mathfrak{E}(J)e_{n-2} = e_{n-2}$, $S(r)e_{n-2} = e_{n-2}$ then there exists a

polyadic ideal which is of order k but is not a Post ideal.

Proof. Let Δ' be a non-trivial proper polyadic ideal of $(B, I, \widehat{\mathbb{E}}, \widehat{S})$ and let $\Delta = \{a : d_k(a) \in \Delta' \text{ and } d_{n-1}(a) = 0\}$. Firstly it shall be demonstrated that Δ is an ideal. Notice that the element $a'e_k = a$ where $a' \in \Delta'$ and $a' \neq 0$ has the property that $d_k(a) \in \Delta'$ and $d_{n-1}(a) = 0$ which results in Δ being non-empty and different from $\{0\}$. Let $a, b \in \Delta$ and considering the element $a \vee b$ we find that $d_k(a \vee b) = d_k(a) \vee d_k(b) \in \Delta'$ and that $d_{n-1}(a \vee b) = d_{n-1}(a) \vee d_{n-1}(b) = 0 \vee 0 = 0$ resulting in $a \vee b \in \Delta$. Let $a \in \Delta$, $c \in P$ and considering the element $a \wedge c$ we find that $d_k(a \wedge c) = d_k(a) \wedge d_k(c) \in \Delta'$ and $d_{n-1}(a \wedge c) = d_{n-1}(a) \wedge d_{n-1}(c) = 0 \wedge d_{n-1}(c) = 0$ results in $a \wedge c \in \Delta$. Δ is, therefore, an ideal. Notice that since e_{k-1} , e_k , e_{n-2} , e_{n-1} are stationary elements for $S(r)$ and $\mathbb{E}(J)$ it follows that $d_k(\mathbb{E}(J)a) = \mathbb{E}(J)d_k(a)$, $d_k(S(r)a) = S(r)d_k(a)$ and that $d_{n-1}(\mathbb{E}(J)a) = \mathbb{E}(J)d_{n-1}(a)$, $d_{n-1}(S(r)a) = S(r)d_{n-1}(a)$. Next it shall be demonstrated that Δ is closed with respect to $\mathbb{E}(J)$ and $S(r)$. Let $a \in \Delta$ and considering the elements $\mathbb{E}(J)a$ and $S(r)a$, we find that $d_k(\mathbb{E}(J)a) = \mathbb{E}(J)d_k(a) \in \Delta'$, $d_k(S(r)a) = S(r)d_k(a) \in \Delta'$ as well as $d_{n-1}(\mathbb{E}(J)a) = \mathbb{E}(J)d_{n-1}(a) = \mathbb{E}(J)0 = 0$, $d_{n-1}(S(r)a) = S(r)d_{n-1}(a) = S(r)0 = 0$. So $S(r)a, \mathbb{E}(J)a \in \Delta$ and Δ is, therefore, a polyadic ideal.

By definition of Δ it is obvious that $e_{k-1} \in \Delta$ and $e_k \notin \Delta$. Assume that Δ is a Post ideal. As $e_{k-1} \in \Delta$ and $e_k \notin \Delta$ for $k \neq n-1$, the ideal Δ is proper and, therefore, of the form $(\Delta \cap B, k)$.

From above it is known that $a'e_k \in \Delta$ where $a' \in \Delta^1$ and $a' \neq 0$. By definition of $(\Delta \cap B, k)$ it follows that $d_k(a'e_k) = a' \in \Delta \cap B$ since $a'e_k \in \Delta$. From the definition of the set Δ , $a' \in \Delta$ implies that $d_{n-1}(a') = 0$ and $a' \in B$ implies that $a' = d_{n-1}(a')$ which is a contradiction. Δ cannot be a Post ideal.

Theorem 4.3. If $(P, I, \mathfrak{E}, S)$ is a polyadic Post algebra where $P = \langle e_0, \dots, e_{n-1} : B \rangle$ and Δ is a proper polyadic ideal of order k where $S(r)e_k \neq e_{k-1}$ for $r \in I^I$ then $(\Delta \cap B, k)$ is a polyadic Post ideal and $(\Delta \cap B, k) \subset \Delta$. (Notice that in the following proof that $(\Delta \cap B, k) \subset \Delta$ the condition of Δ being polyadic is not needed).

Proof. Since Δ and B are closed with respect to $S(r)$ and $\mathfrak{E}(J)$ for $J \subset I$ and $r \in I^I$, $\Delta \cap B$ is closed with respect to these operations. As $e_{k-1} \in \Delta$ and $e_k \notin \Delta$, $\mathfrak{E}(J)e_{k-1} = e_{k-1}$ for $J \subset I$. In particular $\mathfrak{E}(I)e_{k-1} = e_{k-1}$ and as the stationary distinguished elements of $\mathfrak{E}(I)$ are contained in those of $S(r)$ for any $r \in I^I$ $S(r)e_{k-1} = e_{k-1}$ for all $r \in I^I$. (See theorem 3.15.) From this it follows that $d_k(S(r)a) = S(r)d_k(a)$ and $d_k(\mathfrak{E}(J)a) = \mathfrak{E}(J)d_k(a)$ (since $\mathfrak{E}(J)e_k > e_{k-1}$ and $S(r)e_k > e_{k-1}$). Notice that $a \in (\Delta \cap B, k)$ iff $d_k(a) \in \Delta \cap B$. If $d_k(a) \in \Delta \cap B$ then $\mathfrak{E}(J)d_k(a), S(r)d_k(a) \in \Delta \cap B$ giving $d_k(\mathfrak{E}(J)a), d_k(S(r)a) \in \Delta \cap B$ which means that $\mathfrak{E}(J)a$ and $S(r)a$ belong to $(\Delta \cap B, k)$. $(\Delta \cap B, k)$ is, therefore, a polyadic Post ideal of P .

Let $a = \bigcup_{i=1}^{n-1} d_i(a)e_i \in (\Delta \cap B, k)$. Now for all $i \leq k-1$, $e_i \in \Delta$

which implies that $d_i(a)e_i \in \Delta$. By definition of a Post ideal

$d_i(a) \in \Delta \cap B$ for $i \geq k$ which gives for $i \geq k$, $d_i(a) \in \Delta$ resulting in $d_i(a)e_i \in \Delta$. For all i , $d_i(a)e_i \in \Delta$ and hence

$$\bigcup_{i=1}^{n-1} d_i(a)e_i \in \Delta \text{ and } a \in \Delta.$$

Theorem 4.4. If $(P, I, \mathbb{E}, S)$ is a polyadic Post algebra where $P = \langle e_0, \dots, e_{n-1}; B \rangle$ and Δ is a proper polyadic ideal of P of order k where $S(r)e_k \neq e_{k-1}$ for $r \in I^I$, then $\Delta \subset (\{d_k(a) : a \in \Delta\}, k)$ and $(\{d_k(a) : a \in \Delta\}, k)$ is a polyadic Post ideal.

Proof. Firstly, it shall be shown that $\eta = \{d_k(a) : a \in \Delta\}$ is an ideal of B . Obviously $0 \in \eta$, so $\eta \neq \emptyset$. Let $a, b \in \eta$ which implies that there exist $a', b' \in \Delta$ such that $a = d_k(a')$ and $b = d_k(b')$. Now $a' \vee b' \in \Delta$ and $d_k(a' \vee b') = d_k(a') \vee d_k(b') = a \vee b$ which implies that $a \vee b \in \eta$. Let $a \in \eta$ and c be any element of B , and we know that there exists an $a' \in \Delta$ such that $a = d_k(a')$, and that $d_k(c) = c$. Since $a' \wedge c \in \Delta$ and $d_k(a' \wedge c) = d_k(a') \wedge d_k(c) = a \wedge c$ it follows that $a \wedge c \in \eta$ completing the proof that η is an ideal of B . If $a \in \eta$ then $a = d_k(a')$ where $a' \in \Delta$ and Δ being a polyadic ideal implies that $\mathbb{E}(J)a', S(r)a' \in \Delta$ for all $J \subset I$ and $r \in I^I$. Hence $d_k(\mathbb{E}(J)a'), d_k(S(r)a') \in \eta$. Since Δ is of order k it follows by results obtained in the course of the proof of theorem 4.3 that $d_k(\mathbb{E}(J)a') = \mathbb{E}(J)d_k(a') = \mathbb{E}(J)a$, $d_k(S(r)a') = S(r)d_k(a') = S(r)a$ which results in $\mathbb{E}(J)a \in \eta$ and $S(r)a \in \eta$. Now η being a polyadic ideal in $(B, I, \widehat{\mathbb{E}}, \widehat{S})$, (η, k) will be a polyadic Post ideal in P , since $\mathbb{E}(J)d_k(a) = d_k(\mathbb{E}(J)a)$ and $S(r)d_k(a) = d_k(S(r)a)$.

Let $a \in \Delta$. By definition $d_k(a) \in \eta$ and since η is an ideal and since for all $i \geq k$, $d_i(a) \leq d_k(a)$ one has $d_i(a) \in \eta$ for $i \geq k$. Therefore $a \in (\eta, k)$.

(Notice that in the proof that $\Delta \subset \mathfrak{K}(\{d_k(a) : a \in \Delta\}, k)$ the condition of Δ being polyadic and of order k is not needed.)

Theorem 4.5. If Δ is a lattice ideal of P which is of order $n-1$ then Δ is a Post ideal.

Proof. Let $\eta = \{d_{n-1}(a) : a \in \Delta\}$. By theorems 4.3 and 4.4 $(\Delta \cap B, n-1) \subset \Delta \subset (\eta, n-1)$. If it can be shown that $\eta = \Delta \cap B$ then $\Delta = (\eta, n-1)$ and the proof will be completed.

Let $a \in \Delta \cap B$ which implies that $a \in \Delta$ and that $d_{n-1}(a) = a$. By definition of η it is true that $a \in \eta$ and so $\Delta \cap B \subset \eta$. If $a \in \eta$, there exists an $a' \in \Delta$ such that $d_{n-1}(a') = a$. For $0 \leq i \leq n-1$, $d_i(a') \geq d_{n-1}(a') = a$ and as $d_i(a) = a$, $d_i(a') \geq d_i(a)$ results, giving $a' \geq a$. Since $a' \in \Delta$, $a \in \Delta$ and therefore $\eta \subset \Delta \cap B$.

Theorem 4.6. If $(P, I, \mathbb{A}, S)$ is a polyadic Post algebra where $P = \langle e_0, \dots, e_{n-1} : B \rangle$ and Δ is a proper polyadic ideal in P of order k where for all $r \in I^I$, $S(r)e_i \neq e_{i-1}$ for $i \geq k$ then there exists a maximal polyadic ideal which is a Post ideal containing Δ .

Proof. Let $\bar{\Delta}$ be a proper polyadic ideal containing Δ where $\bar{\Delta}$ is of order $\bar{k}$ such that $\bar{k}$ is the maximum of the set of the orders of the proper polyadic ideals containing Δ . Now

$\bar{\Delta} \subset (\{d_{\bar{k}}(a) : a \in \bar{\Delta}\}, \bar{k})$, and $\{d_{\bar{k}}(a) : a \in \bar{\Delta}\}$ and $(\{d_{\bar{k}}(a) : a \in \bar{\Delta}\}, k)$ are polyadic ideals of $(B, I, \hat{H}, \hat{S})$ and $(P, I, \mathfrak{H}, S)$ respectively. Using the fact that each proper polyadic ideal in B can be contained in a maximal polyadic ideal in B (Holmes [6]) $\{d_{\bar{k}}(a) : a \in \bar{\Delta}\}$ is contained in a maximal proper polyadic ideal Δ' in B giving $(\{d_{\bar{k}}(a) : a \in \bar{\Delta}\}, \bar{k}) \subset (\Delta', \bar{k})$.

It will now be demonstrated that $(\Delta', \bar{k})$ is a maximal polyadic ideal of $(P, I, \mathfrak{H}, S)$. Let $(\Delta', \bar{k}) \subset N$ and $N \neq P$ where N is a polyadic ideal in P . From the proof of theorem 4.4, $N \subset (\{d_{\bar{k}}(a) : a \in N\}, \bar{k})$ which gives $(\Delta', \bar{k}) \subset (\{d_{\bar{k}}(a) : a \in N\}, \bar{k})$. So $\Delta' \subset \{d_{\bar{k}}(a) : a \in N\}$. As $\Delta \subset \bar{\Delta} \subset N$ and by the choice of $\bar{k}$ it follows that N is of order $\bar{k}$, from which we can deduce that $\{d_{\bar{k}}(a) : a \in N\} \neq B$. Δ' being a maximal polyadic ideal in B gives $\{d_{\bar{k}}(a) : a \in N\} = \Delta'$. Therefore $(\Delta', \bar{k}) \subset N \subset (\Delta', \bar{k})$ follows and (Δ', k) is a maximal polyadic ideal of $(P, I, \mathfrak{H}, S)$.

2. Polyadic Post Logic.

Definition 4.7.

- I. $(P, I, \mathfrak{H}, S : \Delta)$ is a polyadic Post logic iff
 - 1. $(P, I, \mathfrak{H}, S)$ is a natural polyadic Post algebra, and
 - 2. Δ is a proper polyadic ideal.
- II $(P, I, \mathfrak{H}, S : \Delta)$ is of type (k, n) iff
 - 1. Δ is of order k , and
 - 2. P is of order n .

Definition 4.8.

- I. A model for a polyadic Post logic of type (k, n) is a basically polyadic simple natural polyadic Post algebra of order n , while a model is a basically polyadic simple natural polyadic Post algebra.
- II. An interpretation of a polyadic Post logic $(P, I, E, S : \Delta)$ of type (k, n) is a natural polyadic Post homomorphism h of (P, I, E, S) onto a model $\mathcal{M}$ for $(P, I, E, S : \Delta)$ such that $h(\Delta) \subset (\{0\}, k)_{\mathcal{M}}$.

4.9.

Theorem 4.9. A polyadic Post algebra (P, I, E, S) of infinite degree is a model iff it is isomorphic to a functional polyadic Post algebra whose elements form a Post subalgebra of $\{e_j : j = 0, \dots, n-1\}^{(X^I)}$ where X is a non-empty set.

Proof. Assume firstly that (A, I, E, S) is a functional polyadic Post algebra from a space $\langle e_0, \dots, e_{n-1} : 2 \rangle^{(X^I)}$ where $X \neq \emptyset$.

Let $\hat{A} = A \cap 2^{(X^I)}$ and for all $J \subset I$, $r \in I^I$, $a \in \hat{A}$, $\hat{E}(J)a = E(J)a$ and $\hat{S}(r)a = S(r)a$. From the proof of theorem 3.11 that (A, I, E, S) is a natural polyadic Post algebra, $(\hat{A}, I, \hat{E}, \hat{S})$ will be a polyadic Boolean algebra, because $\hat{A}$ is the set of complemented element of A , (A, I, E, S) will be basically polyadic simple if $(\hat{A}, I, \hat{E}, \hat{S})$ is polyadic simple. From the work of Halmos [6] $(\hat{A}, I, \hat{E}, \hat{S})$ will be polyadic simple if $\hat{E}(I)$ is simple. Now if $p \in \hat{A}$ and $p \neq 0$ then $(\hat{E}(I)p)(y) = \bigcup \{p(x) : x \in X^I \text{ and } xI_* y\} = \bigcup \{p(x) : x \in X^I\} = 1$ for all $y \in X^I$ which means that $\hat{E}(I)$ is simple. It follows, therefore, that (A, I, E, S) and indeed any algebra isomorphic to (A, I, E, S) will be a model.

Assume now that $(P, I, \mathcal{E}, S)$ is a model. Hence $(B, I, \widehat{\mathcal{E}}, \widehat{S})$, where B is the base algebra of P , will be polyadic simple. Now it was shown by A. Daigneault and D. Monk [2] that when I is infinite $(B, I, \widehat{\mathcal{E}}, \widehat{S})$ will be isomorphic to a functional polyadic Boolean algebra whose elements form a Boolean subalgebra B' of $2^{(X^I)}$ where X is a non-empty set. Let $\widehat{h}$ be a polyadic isomorphism of $(B, I, \widehat{\mathcal{E}}, \widehat{S})$ onto a functional polyadic Boolean algebra $(B', I, \mathcal{E}', S')$. Theorem 1.9 of Chapter I shows that P is isomorphic to $P' = \langle E_0, \dots, E_{n-1}; B' \rangle$ where $E_i(x) = e_i$ for all $x \in X^I$, by means of the function h defined as

$$h(a) = \bigcup_{i=1}^{n-1} \widehat{h}(d_i(a))E_i \quad \text{for all } a \in P. \quad P' \text{ is a Post subalgebra of}$$

$$\langle E_0, \dots, E_{n-1}; 2^{(X^I)} \rangle. \quad \text{As } \{e_i; i = 0, \dots, n-1\} \text{ is a complete Post}$$

algebra $(\{e_i; i = 0, \dots, n-1\}^{(X^I)}, I, \mathcal{E}^*, S^*)$ will be a functional polyadic Post algebra where $\mathcal{E}^*$ and S^* are defined as in def. 3.10.

Also from the definition of $\mathcal{E}^*$ and S^* for a functional polyadic

Post algebra it is immediate that $(\widehat{h}(B), I, \mathcal{E}', S') = (h(B), I, \widehat{\mathcal{E}}^*, \widehat{S}^*)$

that is to say $\mathcal{E}' = \widehat{\mathcal{E}}^*$ and $S' = \widehat{S}^*$ where $\widehat{\mathcal{E}}^*(J)$ and $\widehat{S}^*(J)$ are the

restrictions of $\mathcal{E}^*(J)$ and $S^*(J)$ to the base algebra. It then follows

$$\text{that } h(\mathcal{E}(J)a) = h\left(\bigcup_{i=1}^{n-1} (\mathcal{E}(J)d_i(a))e_i\right) = \bigcup_{i=1}^{n-1} h(\mathcal{E}(J)d_i(a))E_i =$$

$$= \bigcup_{i=1}^{n-1} \mathcal{E}'(J)(h(d_i(a)))E_i = \bigcup_{i=1}^{n-1} \widehat{\mathcal{E}}^*(J)(h(d_i(a)))E_i = \mathcal{E}^*(J) \bigcup_{i=1}^{n-1} h(d_i(a))E_i =$$

$$= \mathcal{E}^*(J)h(a), \quad \text{and similarly that } h(S(r)a) = S^*(r)h(a). \quad \text{Then the above}$$

shows that P' is closed with respect to every $\mathcal{E}^*(J)$ and $S^*(r)$ as

well as demonstrating that h is a polyadic isomorphism. Hence every model is isomorphic to a functional polyadic^{Post} algebra whose elements form a Post subalgebra of $\{e_i : i = 0, \dots, n-1\}^{(X^I)}$ where X is a non-empty set.

Definition 4.10.

- I. $(P, I, E, S; \Delta)$ is semantically consistent iff there exists an interpretation of $(P, I, E, S; \Delta)$.
- II. $(P, I, E, S; \Delta)$ of type (k, n) is semantically complete iff for all $a \notin \Delta$ there exists an interpretation h such that $h(a) \notin (\{0\}, k)$.

In the proceeding the semantic consistency of a polyadic Post logic $(P, I, E, S; \Delta)$ shall be demonstrated and a necessary and sufficient condition for $(P, I, E, S; \Delta)$ to be semantically complete will be given. However before this can be accomplished the following lemmas must be considered.

Let Δ be a polyadic ideal of a polyadic Boolean algebra (B, I, E, S) . For $a \in B$ let $[a]$ be the congruence class^{of E and S} modulo Δ . For all $J \subseteq I$, $r \in I^I$, define mappings $E_0(J)$ and $S_0(r)$ on B/Δ , which can be shown to be well defined, by the following $E_0(J)[a] = [E(J)a]$ and $S_0(r)[a] = [S(r)a]$. The following lemma was given by Halmos [6].

Lemma. If (B, I, E, S) is a polyadic Boolean algebra and Δ is a proper polyadic ideal of B , then $(B/\Delta, I, E_0, S_0)$ is a polyadic Boolean algebra and the natural map of B onto B/Δ is a polyadic homomorphism of (B, I, E, S) onto $(B/\Delta, I, E_0, S_0)$.

A polyadic Boolean algebra is said to be semi-simple iff the intersection of the maximal polyadic ideals of the algebra equals $\{0\}$. That a polyadic Boolean algebra is semi-simple was demonstrated by Halmos [6].

Lemma. If $(B, I, \mathbb{E}, S)$ is a polyadic Boolean algebra then for a proper polyadic ideal Δ there exists a set of maximal polyadic ideals $\{\Delta_l : l \in L\}$ such that $\bigcap_{l \in L} \Delta_l = \Delta$.

Proof. Let $h: B \rightarrow B/\Delta$ be the natural map which is a polyadic homomorphism of $(B, I, \mathbb{E}, S)$ onto $(B/\Delta, I, \mathbb{E}_0, S_0)$. $(B/\Delta, I, \mathbb{E}_0, S_0)$ being a polyadic Boolean algebra is semi-simple and the intersection of the set $\{\Delta_l : l \in L\}$ of maximal polyadic ideals of $(B/\Delta, I, \mathbb{E}_0, S_0)$ equals $\{0\}$. Consider now $h^{-1}(\bigcap_{l \in L} \Delta_l) = h^{-1}(\{0\})$ which gives $\bigcap_{l \in L} h^{-1}(\Delta_l) = \Delta$.

Proof of the lemma will, therefore, be complete if it can be shown that $h^{-1}(\Delta_l)$ is a maximal polyadic ideal of $(B, I, \mathbb{E}, S)$. If $a, b \in h^{-1}(\Delta_l)$ then $h(a), h(b) \in \Delta_l$ which gives $h(a) \vee h(b) \in \Delta_l$, and so $h(a \vee b) \in \Delta_l$ and $a \vee b \in h^{-1}(\Delta_l)$. Let $a \in h^{-1}(\Delta_l)$, $c \in B$ and $c \leq a$. Now $h(a) \in \Delta_l$ and $h(c) \in B/\Delta$. Since $c \leq a$, $h(c) \leq h(a)$ which implies that $h(c) \in \Delta_l$ or that $c \in h^{-1}(\Delta_l)$. With $J \subset I$, $r \in I^I$, $a \in h^{-1}(\Delta_l)$ we have since Δ_l is polyadic that $\mathbb{E}_0(J)h(a), S_0(r)h(a) \in \Delta_l$ which means that $h(\mathbb{E}(J)a), h(S(r)a) \in \Delta_l$ giving $\mathbb{E}(J)a, S(r)a \in h^{-1}(\Delta_l)$. $h^{-1}(\Delta_l)$, is therefore, a polyadic ideal of $(B, I, \mathbb{E}, S)$ and it remains to show that it is maximal. Let $\bar{\Delta}$ be a proper polyadic ideal of $(B, I, \mathbb{E}, S)$ containing $h^{-1}(\Delta_l)$. Then $h(\bar{\Delta})$ is a polyadic ideal of

$(B/\Delta, I, \mathbb{E}_0, S_0)$ containing Δ_ℓ . Assume that $[1] \in h(\bar{\Delta})$. Let a be an element of $\bar{\Delta}$ such that $[a] = [1]$. Then $-a \in \Delta$ and $-a \in h^{-1}(\Delta_\ell)$, so $-a \in \bar{\Delta}$ and $1 \in \bar{\Delta}$ giving $\bar{\Delta} = B$, contrary to the hypothesis that $\bar{\Delta}$ is a proper ideal of B . Hence $[1] \notin h(\bar{\Delta})$ and $h(\bar{\Delta})$ is a proper ideal of B/Δ . By the maximality of Δ_ℓ , $h(\bar{\Delta}) = \Delta_\ell$. Hence $\bar{\Delta} \subset h^{-1}(\Delta_\ell)$ and $\bar{\Delta} = h^{-1}(\Delta_\ell)$ results so that $h^{-1}(\Delta_\ell)$ is a maximal polyadic ideal of $(B, I, \mathbb{E}, S)$.

Theorem 4.11. If $(P, I, \mathbb{E}, S; \Delta)$ is a polyadic Post logic then $(P, I, \mathbb{E}, S; \Delta)$ is semantically consistent.

Proof. By theorem 4.4 there exists a polyadic Post ideal Δ' such that $e_{k-1} \in \Delta'$, $e_k \notin \Delta'$ and $\Delta \subset \Delta'$. Let $P = \langle e_0, \dots, e_{n-1} : B \rangle$. Since Δ' is proper $B \cap \Delta'$ is proper and there exists a maximal polyadic ideal M of B containing $\Delta' \cap B$. Let h be the natural map of B onto B/M . h will be a polyadic homomorphism of $(B, I, \hat{\mathbb{E}}, \hat{S})$ onto $(B/M, I, \mathbb{E}_0, S_0)$ if $\mathbb{E}_0$ and S_0 are defined by the identities $\mathbb{E}_0(J)(ha) = h(\mathbb{E}(J)a)$, $S_0(r)(ha) = h(S(r)a)$. As M is maximal $(B/M, I, \mathbb{E}_0, S_0)$ is polyadic simple. Now from the theory of Post algebras there exists a Post algebra $P' = \langle e'_0, \dots, e'_{n-1} : B' \rangle$ where B' is isomorphic to B/M . Let g be an isomorphism of B/M onto B' . Define mappings $\mathbb{E}'(J)$ and $S'(r)$ of B' into B for all $J \subset I$ and $r \in I^I$ by $\mathbb{E}'(J)(g(a)) = g(\mathbb{E}_0(J)a)$ and $S'(r)(g(a)) = g(S(r)a)$ which results in g being a polyadic isomorphism of $(B/M, I, \mathbb{E}_0, S_0)$ onto $(B', I, \mathbb{E}', S')$. Define $h_0 : B \rightarrow B'$ by gh . Therefore h_0 is a polyadic homomorphism of $(B, I, \hat{\mathbb{E}}, \hat{S})$ onto $(B', I, \mathbb{E}', S')$. Extending $\mathbb{E}'(J)$ and $S'(r)$ to mappings $\mathbb{E}^*(J)$ and $S^*(r)$ of P' into P'

by $\mathbb{E}^*(J)a = \bigcup_{i=1}^{n-1} \mathbb{E}'(J)d_i(a)e'_i$ and $S^*(r)a = \bigcup_{i=1}^{n-1} S'(r)d_i(a)e'_i$ gives a

natural polyadic Post algebra $(P', I, \mathbb{E}^*, S^*)$ which is basically polyadic simple and, therefore, a model of our logic $(P, I, \mathbb{E}, S; \Delta)$. The mapping $h_0: B \rightarrow B'$ can be extended in a natural fashion to a mapping of $\langle e_0, \dots, e_{n-1}: B \rangle$ into $\langle e'_0, \dots, e'_{n-1}: B' \rangle$ by

$$f(a) = \bigcup_{i=1}^{n-1} h_0(d_i(a))e'_i \text{ for all } a \in P, \text{ which is obviously a polyadic}$$

homomorphism of $(P, I, \mathbb{E}, S)$ onto $(P', I, \mathbb{E}^*, S^*)$. The four equivalent statements (1) $a \in (M, k)$, (2) $d_i(a) \in M$ for $i \geq k$, (3) $h_0(d_i(a)) = 0$ for $i \geq k$, (4) $f(a) \in (\{0\}, k)$ result in f being an interpretation of the polyadic Post logic $(P, I, \mathbb{E}, S; \Delta)$ since $\Delta \subset \Delta' \subset (M, k)$.

Theorem 4.12. A polyadic Post logic $(P, I, \mathbb{E}, S; \Delta)$ of type (k, n) is semantically complete iff Δ is a Post ideal.

Proof. Assume that Δ is not a Post ideal. Δ is contained in the polyadic Post ideal (η, k) where $\eta = \{d_k(a) : a \in \Delta\}$, and let h be an arbitrary interpretation of the logic $(P, I, \mathbb{E}, S; \Delta)$. If $b \in (\eta, k)$ then $d_k(b) \in \eta$ and there exists an $a \in \Delta$ such that $d_k(a) = d_k(b)$ resulting in $h(d_k(a)) = h(d_k(b))$ which gives $d_k(h(a)) = d_k(h(b))$. Because $a \in \Delta$, $h(a) \in (\{0\}, k)$ which implies that $d_k(h(a)) = 0$ giving $d_k(h(b)) = 0$. Therefore $h(d_k(b)) = d_k(h(b)) = 0$ and $h(b) \in (\{0\}, k)$ follows. Since $\Delta \not\subset (\eta, k)$ there exists an $a \notin \Delta$ such that for all interpretations of $(P, I, \mathbb{E}, S; \Delta)$, $h(a) \in (\{0\}, k)$ which means that $(P, I, \mathbb{E}, S; \Delta)$ is not semantically complete.

Now let Δ be a Post ideal, say $\Delta = \Delta = (\Delta', k)$. $(B, I, \hat{E}, \hat{S})$ being semi-simple and Δ' being a proper polyadic ideal in B , there exists a family of maximal polyadic ideals $\{\Delta_l : l \in L\}$ of $(B, I, \hat{E}, \hat{S})$ such that

$\bigcap_{l \in L} \Delta_l = \Delta'$. Let $a \in P - \Delta$ which implies that $d_k(a) \notin \Delta$. Since

$\bigcap_{l \in L} \Delta_l = \Delta'$ there exists an l such that $d_k(a) \notin \Delta_l$ which gives

$a \notin (\Delta_l, k)$. By a result obtained in the proof of theorem 4.11

(with M replaced by Δ_l) there exists an interpretation f of

$(P, I, \hat{E}, S: \Delta)$ such that $f(a) \notin (\{0\}, k)$. This proves that $(P, I, \hat{E}, S: \Delta)$

is semantically complete.

Corollary. The polyadic Post logics $(P, I, \hat{E}, S: \{0\})$ and $(P, I, \hat{E}, S: \Delta)$ of type $(n-1, n)$ are semantically complete.

Proof. From theorem 4.5 a polyadic ideal of $(P, I, \hat{E}, S)$ of order $n-1$ where P is of order n is a Post ideal. Also $\{0\}$ is obviously a Post ideal.

Because of theorem 4.2 a polyadic Post logic of type (k, n) where $1 \leq k < n-1$ may not be semantically complete.

Ending this chapter, notice that by specializing the set I in the polyadic Post logic $(P, I, \hat{E}, S: \Delta)$ two important special cases may be obtained. For example if $I = \{i\}$ the resulting algebra will be essentially a monadic Post algebra $(P, \hat{E}(\{i\}))$ and the resulting logic will be called a monadic Post logic $(P, \hat{E}(\{i\}): \Delta)$, and if $I = \emptyset$

the resulting algebra will be a Post algebra P and the resulting logic will be called a Post logic $(P;\Delta)$. Hence the monadic Post and the Post logics are special cases of our polyadic Post logic and all the results of this chapter except theorem 4.9 will hold for these special cases. However for the monadic Post logics theorem 2.24 will replace theorem 4.9 so that a model of a monadic Post logic can also be given as a functional monadic Post algebra whose elements form a Post subalgebra of $\{e_i : i = 0, \dots, n-1\}^X$ where X is a non-void set. For a Post logic the basically simple algebras that is to say algebras of the form $\langle e_0, \dots, e_{n-1} : \{0,1\} \rangle$ are the natural models.

REFERENCES

- [1] Church, A., Introduction to Mathematical Logic, Princeton, 1956.
- [2] Daigneault, A. and Monk, D. Representation theory for polyadic algebras, Fund. Math., 52(1963) p 151-176.
- [3] Daigneault, A., On automorphisms of polyadic algebras, Trans. Amer. Math. Soc. 112(1964) p 84-130.
- [4] Dwinger, P., Notes on Post algebras I,II, Proceedings, 69(1966) p 462-478.
- [5] Epstein, G., The lattice theory of Post algebras, Trans. Amer. Math. Soc., 95(1960) p 300-317.
- [6] Halmos, P., Algebraic Logic, Chelsea, 1962.
- [7] Henkin, L. and Tarski, A., Cylindrical algebras, Proc. of Symposia in Pure Math., V 2, Lattice Theory, Amer. Math. Soc., p 83-113.
- [8] Karp, C.R., Languages with expressions of infinite length, North-Holland, 1964.
- [9] Rasiowa, H. and Sikorski, R., The mathematics of metamathematics, Monografie matematyczne, 1963.
- [10] Rosenbloom, P.C., Post algebras I, Postulates and general theory, Amer. Jour. of Math., 64(1942) p 167-185.
- [11] Rosser, J.B., and Turquette, A.R., Many-valued logics, North-Holland, 1952.
- [12] Sikorski, R., Boolean algebras, Springer, 1964.
- [13] Trackzyk, T., Axioms and some properties of Post algebras, Coll. Math., 10(1963) p 193-209.
- [14] Trackzyk, T., A generalization of the Loomis-Sikorski theorem, Coll. Math., 12(1964) p 155-161.

B29880